Maxwell Equation & EM Wave

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What happen ?

When the charge moves with an accelarationatic then

Field – Time varying electric and magnetic field

- Time varying electric and magnetic field : ???
- Variation of E & D and B & H with time : ???
- Differential equation for E & D and B & H : ???
- Magnitude of E & D and B & H : ???
- Direction of E & D and B & H : ???

Maxwell Equation

Mathematical expressions based on the law's of electrostatics, magneto-statics and electromagnetic induction.

Useful in explanation of EM wave or time varying electric and magnetic fields.

First Maxwell Equation

Gauss Law of electrostatics

Net outward electric flux through closed surface =q/ε_0

$$d\phi = \vec{E} \bullet d\vec{s}$$

$$\phi = \oint \vec{E}.d\vec{s}$$

$$dq = \rho dv$$

$$q = \oint \rho dv$$

$$\oint \vec{E}.d\vec{s} = \frac{1}{\varepsilon_0} \oint \rho dv$$

Integral form of Ist Maxwell equation

$$\oint \left(\vec{\nabla}.\vec{E}\right) dv = \frac{1}{\varepsilon_0} \oint \rho dv$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}.\vec{D} = \rho$$

Differential form of Ist Maxwell equation

$$\vec{D} = \varepsilon_0 \vec{E}; \quad \vec{D} = \varepsilon \vec{E}$$

Second Maxwell Equation
Gauss Law of Magnetic field

Monopole can not exists.

Magnetic line of forces are closed curves.

Net outward magnetic =0

$$d\phi = \vec{B} \bullet d\vec{s}$$

$$\phi = \oint \vec{B}.d\vec{s}$$

$$\oint \vec{B}.d\vec{s} = 0$$

Integral form of IInd Maxwell equation

$$\oint \left(\vec{\nabla}.\vec{B}\right) dv = 0$$

$$\vec{\nabla}.\vec{B} = 0$$
$$\vec{\nabla}.\vec{H} = 0$$

Differential form of IInd Maxwell equation

$$\vec{B} = \mu_0 \vec{H}; \quad \vec{B} = \mu \vec{H}$$

Third Maxwell Equation

Faraday's Law of EM Induction

Induced emf = - rate of change of magnetic flux

$$e = -\frac{d\phi}{dt}$$

$$e = \oint \vec{E}.d\vec{l}$$

$$\phi = \oint \vec{B}.d\vec{s}$$

$$\oint \vec{E}.d\vec{l} = -\frac{d}{dt}\oint \vec{B}.d\vec{s}$$

Integral form of IIIrd Maxwell equation

$$\oint \left(\vec{\nabla}\times\vec{E}\right).d\vec{s} = -\oint \frac{d\vec{B}}{dt}.d\vec{s}$$

$$\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{d\vec{H}}{dt}$$

Differential form of IIIrd Maxwell equation

Fourth Maxwell Equation

Modified Ampere's Law by Maxwell

Line integral of magnetic field through closed path=μ_0I

$$\text{line integral of magnetic field} = \oint \vec{B}.d\vec{l}$$

$$dI = \vec{J}.d\vec{s}$$

$$\Rightarrow \oint \vec{B}.d\vec{l} = \mu_0 \oint \vec{J}.d\vec{s}$$

$$\Rightarrow \oint \left(\vec{\nabla} \times \vec{B}\right).d\vec{s} = \mu_0 \oint \vec{J}.d\vec{s}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

This is Differential form of Ampere's Law.

Same result can be also obtained by Biot-Savart Law.

Fourth Maxwell Equation ….contd.

Modified Ampere's Law by Maxwell …..contd

Equation of continuity : Based on conservation of charge

$$\vec{\nabla}.\vec{J} + \frac{d\rho}{dt} = 0$$

J: Current density
ρ: Charge density

$$\because \quad \vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \vec{\nabla} \times \vec{J}$$

$$\therefore \quad \vec{\nabla} \times \vec{J} = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} = 0$$

ρ = constant

So, Ampere's law and Biot savart Law are valid when charge density is constant or charges are moving with constant/uniform.

Fourth Maxwell Equation ….contd.

Modified Ampere's Law by Maxwell …..contd

New current: Due change in electric flux with time

New Current: Displacement Current

$$I_d = \varepsilon_0 \frac{d\phi_e}{dt} = \varepsilon_0 \frac{dEA}{dt} = \varepsilon_0 A \frac{dE}{dt}$$

$$J_d = \frac{I_d}{A} = \varepsilon_0 \frac{dE}{dt} \Rightarrow \vec{J}_d = \varepsilon_0 \frac{d\vec{E}}{dt}$$

$$\vec{J}_d = \varepsilon_0 \frac{d\vec{E}}{dt} = \frac{d\vec{D}}{dt} \qquad \Rightarrow \vec{J}_{total} = \vec{J} + \vec{J}_d$$

$$\therefore \quad \vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \vec{J}_d \right)$$

$$\therefore \quad \vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$$

$$\oint \vec{B}.d\vec{l} = \mu_0 \oint \left(\vec{J} + \vec{J}_d \right) d\vec{s}$$

Integral form and Differential form of Fourth Maxwell equation

Maxwell Equation

Differential form of Maxwell equation

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon} \qquad \vec{\nabla}.\vec{D} = \rho$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \vec{\nabla}.\vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\vec{\nabla} \times \vec{E} = -\mu\frac{d\vec{H}}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu\left(\vec{J} + \vec{J}_d\right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$$

Integral form of Maxwell equation

$$\oint \vec{E}.d\vec{s} = \frac{1}{\varepsilon}\oint \rho dv$$

$$\oint \vec{B}.d\vec{s} = 0$$

$$\oint \vec{E}.d\vec{l} = -\frac{d}{dt}\oint \vec{B}.d\vec{s}$$

$$\oint \vec{B}.d\vec{l} = \mu\oint\left(\vec{J} + \vec{J}_d\right).d\vec{s}$$

Electromagnetic wave
In Free space

Properties of free space

$$\rho = 0;\ \ \sigma = 0;\ \ J = 0;\ \ \mu_r = 1;\ \ \varepsilon_r = 1;\ \ \mu = \mu_0;\ \ \varepsilon = \varepsilon_0$$

Maxwell Equation

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon}$$

$$\vec{\nabla}.\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{d\vec{H}}{dt}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon\frac{d\vec{E}}{dt}$$

Maxwell Equation for free space

$$\vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\mu_0\frac{d\vec{H}}{dt}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon_0\frac{d\vec{E}}{dt}$$

Electromagnetic wave

In Free space: Differential Equation for E & H

Taking curl of IIIrd Maxwell Equation

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\mu_0 \frac{d}{dt}\left(\vec{\nabla}\times\vec{H}\right)$$

$$\vec{\nabla}\left(\vec{\nabla}.\vec{E}\right) - (\vec{\nabla}.\vec{\nabla})\vec{E} = -\mu_0 \frac{d}{dt}\left(\varepsilon_0 \frac{d\vec{E}}{dt}\right)$$

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0 \frac{d^2\vec{E}}{dt^2}$$

$$\nabla^2\vec{E} = \mu_0\varepsilon_0 \frac{d^2\vec{E}}{dt^2}$$

Taking curl of IV[th] Maxwell Equation

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = -\varepsilon_0 \frac{d}{dt}\left(\vec{\nabla}\times\vec{E}\right)$$

$$\vec{\nabla}\left(\vec{\nabla}.\vec{H}\right) - (\vec{\nabla}.\vec{\nabla})\vec{H} = -\varepsilon_0 \frac{d}{dt}\left(\mu_0 \frac{d\vec{H}}{dt}\right)$$

$$-\nabla^2\vec{H} = -\mu_0\varepsilon_0 \frac{d^2\vec{H}}{dt^2}$$

$$\nabla^2\vec{H} = \mu_0\varepsilon_0 \frac{d^2\vec{H}}{dt^2}$$

Progresive wave Equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{d^2\psi}{dt^2}$$

E and H are progresive wave

$$\vec{E}(r,t) = \vec{E}_0\, e^{j\left(\vec{K}.\vec{r}\text{-}\omega t\right)}$$

$$\vec{H}(r,t) = \vec{H}_0\, e^{j\left(\vec{K}.\vec{r}\text{-}\omega t\right)}$$

Electromagnetic wave

In Free space: Nature of E and H

$$\vec{\nabla}.\vec{E} = 0 \Rightarrow \vec{K}.\vec{E} = 0 \Rightarrow \vec{K} \perp \vec{E}$$

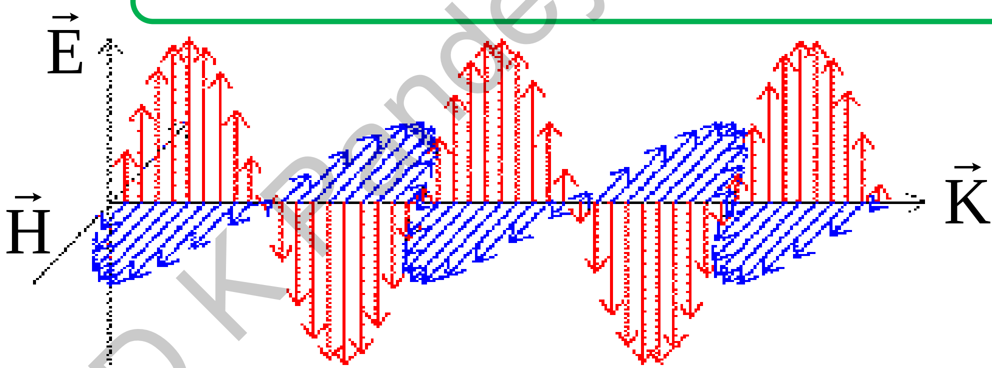
$$\vec{\nabla}.\vec{H} = 0 \Rightarrow \vec{K}.\vec{H} = 0 \Rightarrow \vec{K} \perp \vec{H}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{d\vec{H}}{dt} \Rightarrow \vec{K} \times \vec{E} = \omega\mu_0 \vec{H} \Rightarrow \vec{H} \perp \vec{K} \, \& \, \vec{E}$$

$$\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{d\vec{E}}{dt} \Rightarrow \vec{K} \times \vec{H} = -\varepsilon_0 \omega \vec{E} \Rightarrow \vec{E} \perp \vec{K} \, \& \, \vec{H}$$

⇒ $\vec{E}$, $\vec{H}$ and $\vec{K}$ are mutually perpendicular.

$$\vec{K} = \frac{2\pi}{\lambda} \vec{n}$$



$$\vec{E}_x = \vec{E}_0 \, e^{(k_z z - \omega t)}$$
$$\vec{H}_Y = \vec{H}_0 \, e^{(k_z z - \omega t)}$$

Electromagnetic wave

In Free space: velocity of em wave

$$\nabla^2\vec{E} = \mu_0\varepsilon_0 \frac{d^2\vec{E}}{dt^2}$$

$$\nabla^2\vec{H} = \mu_0\varepsilon_0 \frac{d^2\vec{H}}{dt^2}$$

Comparing

$$\nabla^2\psi = \frac{1}{v^2}\frac{d^2\psi}{dt^2}$$

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \sqrt{\frac{4\pi}{\mu_0} \times \frac{1}{4\pi\varepsilon_0}}$$

$$v = \sqrt{10^7 \times 9 \times 10^9} = 3\times10^8 = C$$

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = C$$

Electromagnetic wave

Impedance of Free space for em wave

$$Z_0 = \text{Resistance offered by free space medium}$$

$$Z_0 = \frac{|\vec{E}|}{|\vec{H}|} = \frac{|\vec{E}_0|}{|\vec{H}_0|}$$

$$\because \quad \vec{K} \times \vec{E} = \omega \mu_0 \vec{H}$$

$$\therefore \frac{E}{H} = \mu_0 \frac{\omega}{K} = \mu_0 v = \mu_0 c$$

$$Z_0 = \mu_0 c$$

$$Z_0 = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \ \Omega \qquad \because v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$$

Electromagnetic wave

Energy density of EM wave in Free space

$$u = \text{Energy stored in unit volume}$$

$$u = u_e + u_m$$

$$\because\ u_e = \frac{1}{2}\varepsilon_0 E^2 \quad \& \quad u_m = \frac{1}{2}\mu_0 H^2$$

$$\therefore\ \frac{u_e}{u_m} = \frac{\varepsilon_0 E^2}{\mu_0 H^2} = \frac{\varepsilon_0 \mu_0}{\mu_0 \varepsilon_0} = 1 \quad \Rightarrow \boxed{u_e = u_m}$$

$$\boxed{u = u_e + u_e = 2u_e = \varepsilon_0 E^2}$$

$$\boxed{\langle u \rangle = \varepsilon_0 \langle E^2 \rangle = \frac{1}{2}\varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2}$$

Electromagnetic wave

Poynting Vector in Free space

$\vec{S}$ = Energy flowing per unit area per unit time along $\vec{K}$

$\vec{S}$ = Power crossing per unit area along $\vec{K}$

$$\vec{S} = \vec{E} \times \vec{H}$$

$$\vec{S} = \vec{E} \times \frac{\vec{K} \times E}{\mu_0 \omega} = \vec{E} \times \frac{\vec{n} \times \vec{E}}{\mu_0 c}$$

$$\vec{S} = \frac{E^2}{\mu_0 c} \vec{n} = \frac{E^2}{Z_0} \vec{n}$$

$$\langle \vec{S} \rangle = \frac{E_{rms}^2}{\mu_0 c} \vec{n} = \frac{E_{rms}^2}{Z_0} \vec{n}$$

$$\langle \vec{S} \rangle = \langle u \rangle c \, \vec{n} \quad \Rightarrow \text{energy flux} = \text{energy density} \times c$$

Electromagnetic wave

Poynting Theorem

$\vec{S}$ = Energy flowing per unit area per unit time along $\vec{K}$

$\vec{S}$ = Power crossing per unit area $\vec{K}$

$$-\vec{J}.\vec{E} = \frac{\partial u}{\partial t} + \vec{\nabla}.\vec{S}$$

- $-\vec{J}.\vec{E}$ = rate of energy transfered into em wave
  $-\vec{J}.\vec{E}$ = Power transfered into em field/wave
- $\frac{\partial u}{\partial t}$ = rate of change of electromagnetic energy
- $\vec{\nabla}.\vec{S}$ = energy flowing out through the boundary surface

A Lot of Thanks

for kind attention