Lecture 9,10&11: Circuit Analysis and theorems

Note A: Some definitions about circuit

Circuit component: The resistor, capacitor, inductor and dc/ac energy sources are called as circuit component. There are two types of circuit components.

Passive component: The electrical component through which energy is dissipated are called as passive component. e.g. resistor, inductor, and capacitor.

Active component: The circuit component which generates energy is called as active component. e.g. dc or ac sources.

Branch: Series combination of circuit component having two terminals is called branch of electrical network. The current through branch components is same.

Node: The junction point of two or more branches in an electrical network is called node.

Electrical network/circuit: The interconnection of electrical circuit components (resistors, capacitors, inductors and energy sources) which results a closed path is called as electrical network.

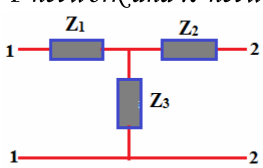
Active network: An electrical circuit containing both the active and passive components is called as active network.

Passive network: An electrical circuit containing only passive components is called as passive network.

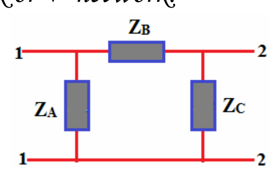
Linear network: If current in electrical circuit is directly proportional to the source voltage then the network is termed as linear network. i.e. there is linear relationship between current and voltage for this network.

Non-linear network: If current in electrical circuit is not directly proportional to the source voltage then the network is termed as non-linear network. i.e. there is non-linear relationship between current and voltage for this network.

Four terminal network: If an electrical network has two input and two output terminal then it is called as four terminal network. e.g. T-network or Y-network and π-network or ∇-network.



T or Y network



π or ∇ network

Loop: Any closed path in electrical network is called loop. On the basis of number of loops, the electrical network may be single, double or multiple loop network.

Mesh: The smallest loop in an electrical network is called as mesh. No closed path can be formed inside a mesh. A mesh is always a loop but all loop can not be called a mesh.

Conversion of T to π network: If Z_1, Z_2 and Z_3 impedances of T-network then impedances of π-network can be calculated with following formulas.

$$Z_A = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_B = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

$$Z_C = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

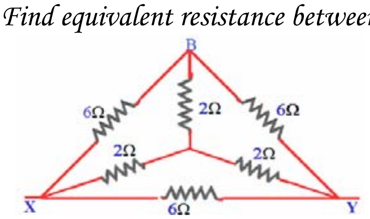
Conversion of π to T network: If Z_A, Z_B and Z_C impedances of π-network then impedances of T-network can be calculated with following formulas.

$$Z_1 = \frac{Z_AZ_B}{Z_A + Z_B + Z_C}$$

$$Z_2 = \frac{Z_BZ_C}{Z_A + Z_B + Z_C}$$

$$Z_3 = \frac{Z_CZ_A}{Z_A + Z_B + Z_C}$$

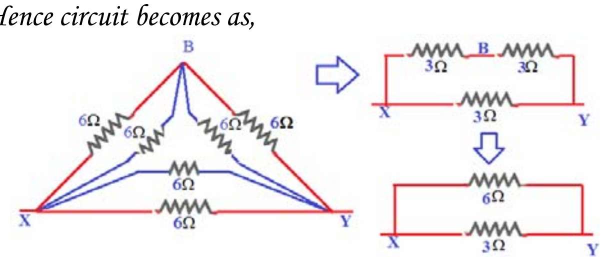
Example 1: Find equivalent resistance between XY.



Solution: This network contains a T-network inside the big triangle. Here, $Z_1=Z_2=Z_3=2\Omega$. Thus

$$Z_A = Z_B = Z_C = \frac{2X2 + 2X2 + 2X2}{2} = \frac{12}{2} = 6\,\Omega$$

Hence circuit becomes as,



From the above resolved circuits, it is clear that the equivalent resistance between XY is resultant of 3 and 6 in parallel combination. R_{XY}=(6X3)/(6+3)=18/9=2Ω

Note B: Loop impedance: *The total impedance of a mesh/loop is called as loop impedance. If a network has two meshes then Z_{11} and Z_{22} are called as loop impedance of mesh first and second respectively.*

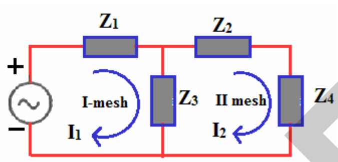
Mutual impedance: *The mutual impedance between the two loops are the ratio of voltage induced in the second loop by the current flowing in the first loop and current in the first loop while all other loops are open circuited.*

Mut. Impd. between 1 & 2$=Z_{12} = -I_1 Z_3 / I_1 = -Z_3$

Mut. Impd. between 2 & 1$=Z_{21} = -I_2 Z_3 / I_2 = -Z_3$

or

The common impedance between two meshes whose polarity depends on direction mesh currents is called as mutual impedance. If direction of both mesh current for common impedance is same then mutual impedance has positive value of impedance. If the currents are opposite in direction then it has negative value. If a network has two meshes then Z_{12} and Z_{21} are called as mutual impedance of between meshes 1 and 2.



For the above two mesh network,

$Z_{11}=Z_1+Z_3$, $\quad Z_{12} = -Z_3$

$Z_{22}=Z_2+Z_3+Z_4$, $\quad Z_{21} = -Z_3$

Here the mutual impedance has negative value because the current flowed by both mesh current in common impedance is opposite in direction.

Note C: Kirchoff's Current Law (KCL): *The algebraic sum of currents at node in electrical network is equal to zero.*

$$\sum i = 0\Omega$$

Incoming current –Out going current=0

Incoming current =Out going current

Note D: Kirchoff's Vltage law (KVL): *The algebraic sum of instantaneous voltage drop across the circuit elements for a closed loop is zero.*

$$\sum V = 0$$

$$\sum iR - \sum E = 0$$

$$\sum iR = \sum E$$

Or $$\sum iZ = \sum E$$

Note E: Mesh Analysis: *Mesh analysis is a method that is used to solve planar circuits for the currents (and indirectly the voltages) at any place in the circuit. Planar circuits are circuits that can be drawn on a plane surface with no wires crossing each other. Mesh analysis and loop analysis both make use of Kirchhoff's voltage law to arrive at a set of equations. Mesh analysis is usually easier to use when the circuit is planar, compared to loop analysis.*

Method:

1. Draw the current in each mesh.

2. Write down the mesh equations in terms of mesh current, loop impedance, mutual impedance and e.m.f. of energy sources used in mesh. For two mesh network, the equations can be written as,

$$Z_{11} I_1 + Z_{12} I_2 = E_1$$

$$Z_{21} I_1 + Z_{22} I_2 = E_2$$

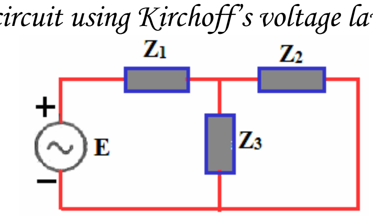
Here E_1 and E_2 are algebraic sum of e.m.f. of energy sources in mesh 1 and 2 respectively. If arrow of drawn current reaches at negative terminal of sources then it is taken as positive and if it reaches at positive terminal then it is taken as negative.

3. Solve the mesh equations for the mesh currents by the matrix method. For the two mesh network, the current equations in terms of matrix can be written as,

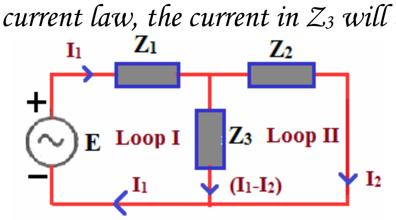
$$I_1 = \frac{\begin{bmatrix} E_1 & Z_{12} \\ E_2 & Z_{22} \end{bmatrix}}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} \text{ and } I_2 = \frac{\begin{bmatrix} Z_{11} & E_1 \\ Z_{21} & E_2 \end{bmatrix}}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}$$

Using the mesh currents, the current or voltage in each component can be determined.

Example2: *Find the current in Z_1, Z_2 and Z_3 in following circuit using Kirchoff's voltage law.*



Solution: The given circuit is two loop network. Let I_1 and I_2 are current in Z_1 and Z_2 respectively. Then from Kirchoff's current law, the current in Z_3 will be (I_1-I_2).



Applying KVL for loop I, we have,

$$Z_1 I_1 + Z_3(I_1 - I_2) = E$$

$$(Z_1 + Z_3)I_1 - Z_3 I_2 = E \qquad (1)$$

Applying KVL for loop II, we have,

$$Z_2 I_2 - Z_3(I_1 - I_2) = 0$$

$$-Z_3 I_1 + (Z_2 + Z_3)I_2 = 0 \qquad (2)$$

By doing, {Eq.(1) X(Z_2+Z_3) }+ {Eq.(2)XZ_3} we have

$$(Z_1 + Z3)(Z_2 + Z_3)I_1 - Z_3^2 I_1 = E(Z_2 + Z_3)$$

$$I_1 = \frac{E(Z_2 + Z_3)}{\left((Z_1 + Z3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$I_1 = \frac{E(Z_2 + Z_3)}{\left(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1 + Z_3^2 - Z_3^2\right)}$$

$$\boxed{I_1 = \frac{E(Z_2 + Z_3)}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (3)$$

By doing, {Eq.(1) X Z_3 }+ {Eq.(2)X(Z_1+Z_3)} we have

$$-Z_3^2 I_2 + (Z_1 + Z3)(Z_2 + Z_3)I_2 = E\, Z_3$$

$$I_2 = \frac{Z_3 E}{\left((Z_1 + Z3)(Z_2 + Z_3) - Z_3^2\right)}$$

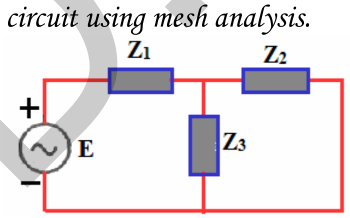
$$\boxed{I_2 = \frac{Z_3 E}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (4)$$

So the current in Z_3= I_1-I_2

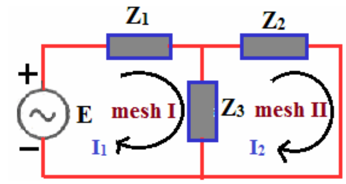
$$I_1 - I_2 = \frac{(Z_2 + Z_3)E - Z_3 E}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}$$

$$\boxed{I_1 - I_2 = \frac{Z_2 E}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (5)$$

Example3: Find the current in Z_1, Z_2 and Z_3 in following circuit using mesh analysis.



Solution: The given circuit is two mesh network. Let I_1 and I_2 are currents in mesh I and mesh II respectively. The direction of currents are shown in Fig(ZZ).



Fig(ZZ)

The mesh equation for mesh I and mesh II can be written as,

$$Z_{11} I_1 + Z_{12} I_2 = E_1 \qquad (1)$$

$$Z_{21} I_1 + Z_{22} I_2 = E_2 \qquad (2)$$

Here, $Z_{11} = Z_1 + Z_3$; $Z_{12} = -Z_3$

$Z_{21} = -Z_3$; $Z_{22} = Z_2 + Z_3$

$E_1 = E$; $E_2 = 0$

So, the currents I_1 and I_2 will be obtained as,

$$I_1 = \frac{\begin{bmatrix} E_1 & Z_{12} \\ E_2 & Z_{22} \end{bmatrix}}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \frac{\begin{bmatrix} E & -Z_3 \\ 0 & Z_2 + Z_3 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_1 = \frac{E(Z_2 + Z_3) - 0}{\left((Z_1 + Z3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$I_1 = \frac{E(Z_2 + Z_3)}{\left(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1 + Z_3^2 - Z_3^2\right)}$$

$$\boxed{I_1 = \frac{E(Z_2 + Z_3)}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (3)$$

And, $$I_2 = \frac{\begin{bmatrix} Z_{11} & E_1 \\ Z_{21} & E_2 \end{bmatrix}}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \frac{\begin{bmatrix} Z_1 + Z_3 & E \\ -Z_3 & 0 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_2 = \frac{0 - (-Z_3)E}{\left((Z_1 + Z3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I_2 = \frac{Z_3 E}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (4)$$

On the basis of current directions, the current in Z_1, Z_2 and Z_3 will be I_1, I_2 and (I_1-I_2) respectively.

So the current in Z_3= I_1-I_2

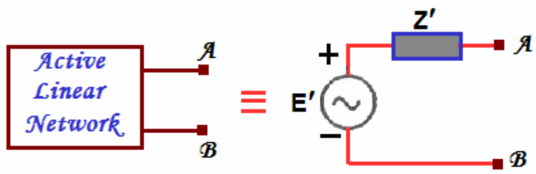
$$I_1 - I_2 = \frac{(Z_2 + Z_3)E - Z_3 E}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}$$

$$\boxed{I_1 - I_2 = \frac{Z_2 E}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (5)$$

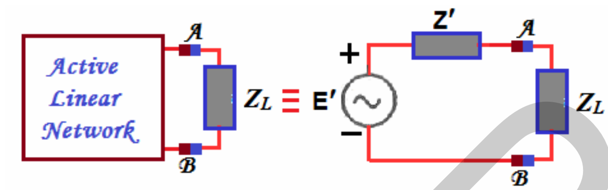
Note: It is clear from example2 and example3 that both the KVl and mesh analysis provides same result.

Thevenin Theorem: *This theorem simplifies the complex active network.*

Statement: *The theorem states that "Any two terminal linear network containing energy sources and impedances (active network) is equivalent to a voltage source of E' in combination with impedance Z' in series. Where E' is open circuited voltage across the terminals of network and Z' is the impedance of network when the sources are replaced by their internal impedances or short circuited.*

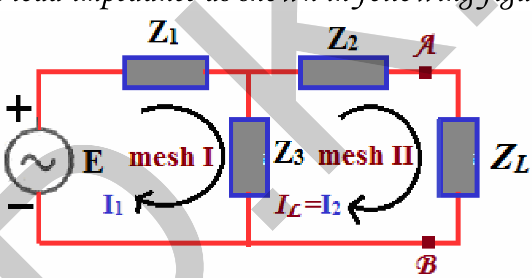


If a load impedance Z_L *is connected to the terminal A and B of the network then load current* I_L *can be written as,*



$$\boxed{I_L = \frac{E'}{Z' + Z_L}}$$

Proof: *let a T-network is connected with a source and load impedance as shown in following figure.*



This network has two meshes. Let the mesh current in first and second meshes are I_1 *and* $I_2 = I_L$ *respectively. The mesh equation for mesh-I and mesh II can be written as,*

$$Z_{11}I_1 + Z_{12}I_2 = E_1 \qquad (1)$$

$$Z_{21}I_1 + Z_{22}I_2 = E_2 \qquad (2)$$

Here,

$$Z_{11} = Z_1 + Z_3\,; \quad Z_{12} = -Z_3$$
$$Z_{21} = -Z_3 \quad ; \quad Z_{22} = Z_2 + Z_3 + Z_L$$
$$E_1 = E \quad ; \quad E_2 = 0$$

Using eqs.(1) and (2), the current in mesh second can be written as,

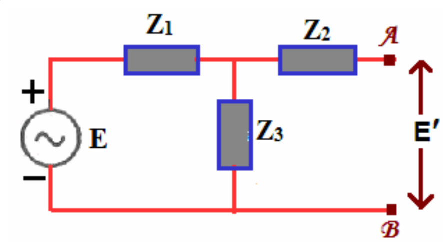
$$I_2 = \frac{\begin{bmatrix} Z_{11} & E_1 \\ Z_{21} & E_2 \end{bmatrix}}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \frac{\begin{bmatrix} Z_1 + Z_3 & E \\ -Z_3 & 0 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 + Z_L \end{bmatrix}}$$

$$I_2 = I_L = \frac{0 - (-Z_3)E}{\left\{(Z_1 + Z_3)(Z_2 + Z_3 + Z_L) - Z_3^2\right\}}$$

$$I_L = \frac{Z_3 E}{\left\{(Z_1 + Z_3)Z_2 + Z_1 Z_3 + (Z_1 + Z_3)Z_L\right\}}$$

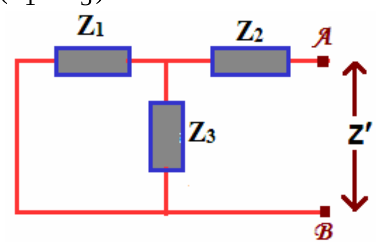
$$I_L = \frac{\frac{Z_3}{(Z_1 + Z_3)}E}{\left\{\left(Z_2 + \frac{Z_1 Z_3}{(Z_1 + Z_3)}\right) + Z_L\right\}} \qquad (4)$$

If E' is open circuited voltage across the terminals of network and Z' is the impedance of network when the sources are short circuited then,



$$E' = \textit{Voltage across } Z_3$$

$$E' = \frac{Z_3}{(Z_1 + Z_3)}E \qquad (5)$$



And

$$Z' = Z_2 + (Z_1 \parallel Z_3)$$

$$Z' = Z_2 + \frac{Z_1 Z_3}{(Z_1 + Z_3)} \qquad (6)$$

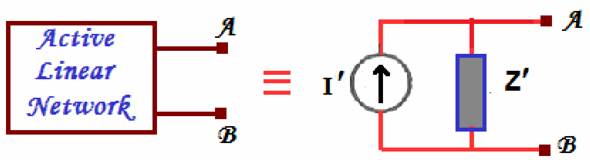
Putting values of E' and Z' in eq.(4), we have,

$$I_L = \frac{E'}{Z' + Z_L} \qquad (7)$$

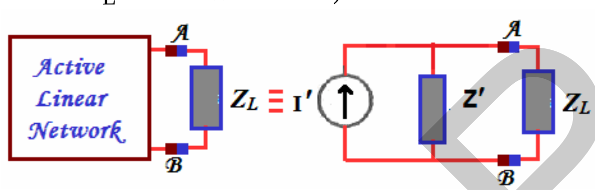
Hence the theorem is proved.

Norton's Theorem: *This theorem simplifies the complex active network by introducing a current source and equivalent impedance.*

Statement: *The theorem states that "Any two terminal linear network containing energy sources and impedances (active network) is equivalent to a current source of I' in parallel combination with impedance Z'. Where I' is short circuited current through terminals of network and Z' is the impedance of network when the sources are replaced by their internal impedances or short circuited.*

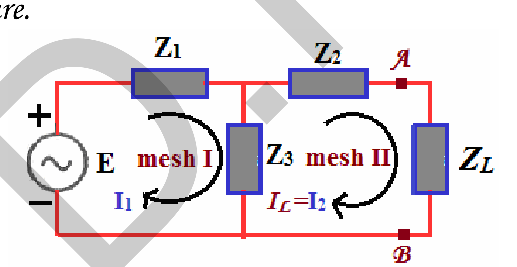


If a load impedance Z_L is connected to the terminal A and B of the network then load current I_L can be written as,



$$\boxed{I_L = \frac{Z'I'}{Z' + Z_L} = \frac{I'}{1 + \dfrac{Z_L}{Z'}}}$$

Proof: *Let a T-network is connected with a source and load impedance as shown in following figure.*



This network has two meshes. Let the mesh current in first and second meshes are I_1 and $I_2 = I_L$ respectively. The mesh equation for mesh-I and mesh II can be written as,

$$Z_{11}I_1 + Z_{12}I_2 = E_1 \qquad (1)$$

$$Z_{21}I_1 + Z_{22}I_2 = E_2 \qquad (2)$$

Here, $Z_{11} = Z_1 + Z_3\,;\ Z_{12} = -Z_3$

$Z_{21} = -Z_3\ ;\ Z_{22} = Z_2 + Z_3 + Z_L$

$E_1 = E\ ;\ E_2 = 0$

Using eqs.(1) and (2), the current in mesh second can be written as,

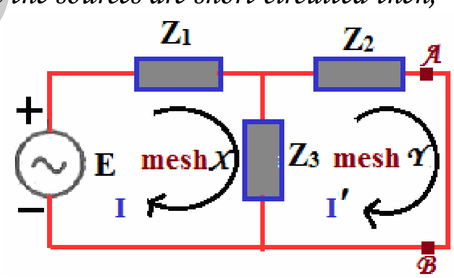
$$I_2 = \frac{\begin{bmatrix} Z_{11} & E_1 \\ Z_{21} & E_2 \end{bmatrix}}{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}} = \frac{\begin{bmatrix} Z_1 + Z_3 & E \\ -Z_3 & 0 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 + Z_L \end{bmatrix}}$$

$$I_2 = I_L = \frac{0 - (-Z_3)E}{\left\{(Z_1 + Z_3)(Z_2 + Z_3 + Z_L) - Z_3^2\right\}}$$

$$I_L = \frac{Z_3E}{\{Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + (Z_1 + Z_3)Z_L\}}$$

$$I_L = \frac{\dfrac{Z_3E}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}}{\left\{1 + \dfrac{Z_1 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} Z_L\right\}} \qquad (4)$$

If I' is short circuited current through terminals of network and Z' is the impedance of network when the sources are short circuited then,



The mesh equation for mesh X and mesh Y can be written as,

$$(Z_1 + Z_3)I - Z_3I' = E \qquad (5)$$

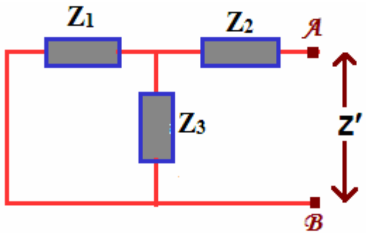
$$-Z_3I + (Z_2 + Z_3)I' = 0 \qquad (6)$$

From eqs.(5) and (6), we can write,

$$I' = \frac{\begin{bmatrix} Z_1 + Z_3 & E \\ -Z_3 & 0 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I' = \frac{0 - (-Z_3)E}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I' = \frac{Z_3E}{(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)}} \qquad (7)$$



And $Z' = Z_2 + (Z_1 \parallel Z_3)$

$$Z' = Z_2 + \frac{Z_1 Z_3}{(Z_1 + Z_3)}$$

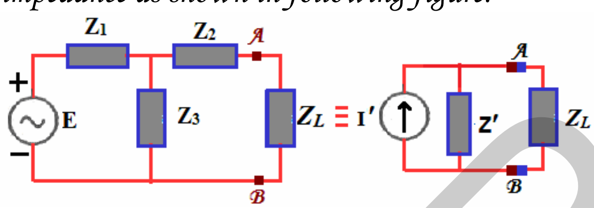
$$Z' = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1 + Z3} \qquad (8)$$

Putting values of I' and Z' in eq.(4), we have,

$$I_L = \frac{I'}{1 + \frac{Z_L}{Z'}} = \frac{Z'I'}{Z' + Z_L} \qquad (9)$$

Hence the theorem is proved.

Alternate Proof of Norton's Theorem: *Let a T-network is connected with a source and load impedance as shown in following figure.*



If I_L is current through lad, then Voltage across load=$V_L = Z_L I_L$

So, $$I_L = \frac{V_L}{Z_L} \qquad (1)$$

Since, V_L = net voltage applied to the parallel combination of Z' and Z_L

i.e. $V_L = Z\,I'$

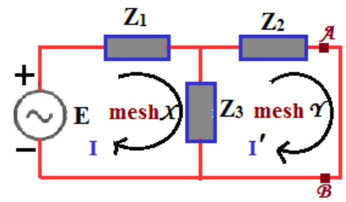
where, $Z = Z' \parallel Z_L = \dfrac{Z'\,Z_L}{Z' + Z_L}$

hence, eq.(1) becomes as,

$$I_L = \frac{Z\,I'}{Z_L} = \frac{Z'\,Z_L}{Z' + Z_L} \times \frac{I'}{Z_L}$$

$$\boxed{I_L = \frac{Z'I'}{Z' + Z_L} = \frac{I'}{1 + \frac{Z_L}{Z'}}} \qquad (2)$$

The short circuited current through terminals of network (I') and equivalent impedance of network (Z') can be determined as,



The mesh equation for mesh X and mesh Y can be written as,

$$(Z_1 + Z_3)I - Z_3 I' = E \qquad (3)$$

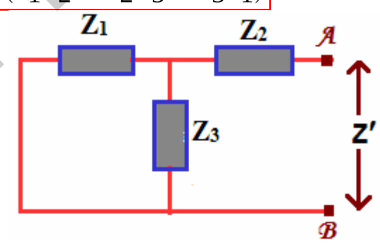
$$-Z_3 I + (Z_2 + Z_3)I' = 0 \qquad (4)$$

From eqs.(3) and (4), we can write,

$$I' = \frac{\begin{bmatrix} Z_1 + Z_3 & E \\ -Z_3 & 0 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I' = \frac{0 - (-Z_3)E}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I' = \frac{Z_3 E}{(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)}} \qquad (5)$$



And $Z' = Z_2 + (Z_1 \parallel Z_3)$

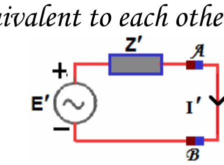
$$\boxed{Z' = Z_2 + \frac{Z_1 Z_3}{(Z_1 + Z_3)}}$$

Or $$\boxed{Z' = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1 + Z3}} \qquad (6)$$

Equivalence of Thevenin and Norton's Theorem

From Thevenin theorem, the load current can be written as, $I_L = \dfrac{E'}{Z' + Z_L}$

If I' is shot circuited current through terminal of network, then $E' = Z'I'$ hence, $I_L = \dfrac{Z'I'}{Z' + Z_L}$. This is Nortons formula for I_L. Therefore both the theorem are equivalent to each other.



Lecture 9,10&11: Circuit Analysis and theorems

Superposition Theorem: .

Statement: *The theorem states that "In any linear network containing impedances and energy sources (active linear network), the current in any element or branch or mesh is equal to algebraic sum of currents that would separately flow in that by each source while other sources are replaced by their internal impedances ".*

$$I = \sum I_x \quad ; x = 1,2,\ldots,n$$

Here n is number of energy sources in the network. If a two mesh active linear network has two energy sources and I_1 & I_2 are the currents in mesh1 and mesh 2 then,

$$I_1 = I_1' + I_1''$$
$$I_2 = I_2' + I_2''$$

Where I_1' and I_2' are currents in mesh1 and mesh 2 by first energy source while I_1'' and I_2'' are currents by second energy source flowing in same direction as I_1 and I_2.

Proof: *Suppose the active linear network is composed of two meshes and has two energy sources as shown in Figure (1).*

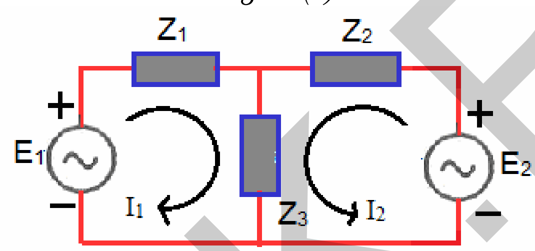


Figure (1)

Let the net current in mesh 1 and mesh 2 are I_1 & I_2 then the mesh equations for the both meshes can be written as,

$$(Z_1 + Z_3)I_1 + Z_3 I_2 = E_1 \qquad (1)$$
$$Z_3 I_1 + (Z_2 + Z_3)I_2 = E_2 \qquad (2)$$

If I_1' and I_2' are currents in mesh1 and mesh2 by first energy source flowing in same direction as I_1 and I_2 {Figure (2)} then mesh equation are written as.

$$(Z_1 + Z_3)I_1' + Z_3 I_2' = E_1 \qquad (3)$$
$$Z_3 I_1' + (Z_2 + Z_3)I_2' = 0 \qquad (4)$$

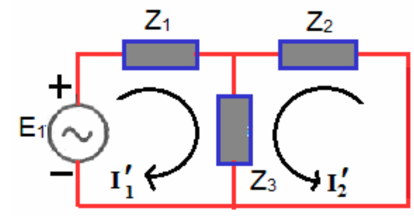


Figure (2)

Similarly if I_1'' and I_2'' are currents in mesh1 and mesh2 by second energy source flowing in same direction as I_1 and I_2 {Figure (3)} then mesh equation are written as.

$$(Z_1 + Z_3)I_1'' + Z_3 I_2'' = 0 \qquad (5)$$
$$Z_3 I_1'' + (Z_2 + Z_3)I_2'' = E_2 \qquad (6)$$

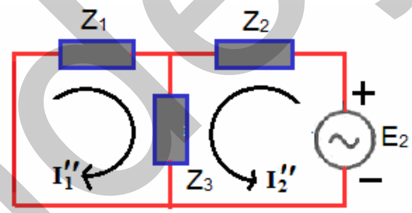


Figure (3)

Doing eq.(3)+ eq.(5) and eq.(2)+ eq.(6), we have,

$$(Z_1 + Z_3)(I_1' + I_1'') + Z_3(I_2' + I_2'') = E_1 \quad (7)$$
$$Z_3(I_1' + I_1'') + (Z_2 + Z_3)(I_2' + I_2'') = E_2 \quad (8)$$

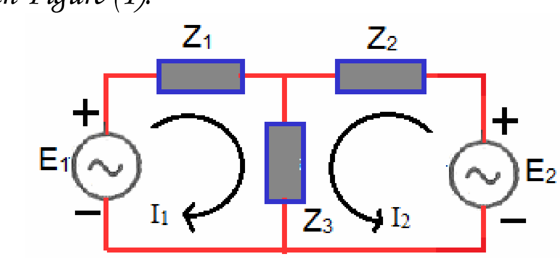
Eqs.(7) & (8) are similar to eqs.(1) & (2), such that,

$$I_1 = I_1' + I_1''$$
$$I_2 = I_2' + I_2''$$

Thus the theorem is proved.

Alternate Proof of Superposition theorem

Suppose the active linear network is composed of two meshes and has two energy sources as shown in Figure (1).



Let the net current in mesh 1 and mesh 2 are I_1 & I_2 then the mesh equations for the both meshes can be written as,

$$(Z_1 + Z_3)I_1 + Z_3 I_2 = E_1 \qquad (1)$$
$$Z_3 I_1 + (Z_2 + Z_3)I_2 = E_2 \qquad (2)$$

From eqs.(1) and (2), we can write,

$$I_1 = \frac{\begin{bmatrix} E_1 & Z_3 \\ E_2 & Z_2 + Z_3 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_1 = \frac{(Z_2 + Z_3)E_1 - Z_3 E_2}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I_1 = \frac{(Z_2 + Z_3)E_1}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)} + \frac{-Z_3 E_2}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (3)$$

And,

$$I_2 = \frac{\begin{bmatrix} Z_1 + Z_3 & E_1 \\ Z_3 & E_2 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_2 = \frac{(Z_1 + Z_3)E_2 - Z_3 E_1}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I_2 = \frac{-Z_3 E_1}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)} + \frac{(Z_1 + Z_3)E_2}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (4)$$

If I_1' and I_2' are currents in mesh1 and mesh2 by first energy source flowing in same direction as I_1 and I_2 {Figure (2)} then mesh equation are written as.

$$(Z_1 + Z_3)I_1' + Z_3 I_2' = E_1 \qquad (5)$$

$$Z_3 I_1' + (Z_2 + Z_3)I_2' = 0 \qquad (6)$$

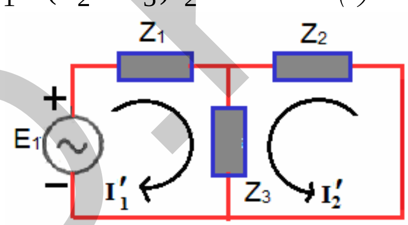


Figure (2)

From eqs.(5) and (6) we can write,

$$I_1' = \frac{\begin{bmatrix} E_1 & Z_3 \\ 0 & Z_2 + Z_3 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_1' = \frac{(Z_2 + Z_3)E_1 - 0}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I_1' = \frac{(Z_2 + Z_3)E_1}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (7)$$

And,

$$I_2' = \frac{\begin{bmatrix} Z_1 + Z_3 & E_1 \\ Z_3 & 0 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_2' = \frac{0 - Z_3 E_1}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I_2' = \frac{-Z_3 E_1}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (8)$$

Similarly if I_1'' and I_2'' are currents in mesh1 and mesh2 by second energy source flowing in same direction as I_1 and I_2 {Figure (3)} then mesh equation are written as.

$$(Z_1 + Z_3)I_1'' + Z_3 I_2'' = 0 \qquad (9)$$

$$Z_3 I_1'' + (Z_2 + Z_3)I_2'' = E_2 \qquad (10)$$

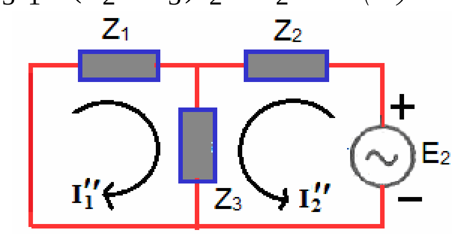


Figure (3)

From eqs.(9) and (10) we can write,

$$I_1'' = \frac{\begin{bmatrix} 0 & Z_3 \\ E_2 & Z_2 + Z_3 \end{bmatrix}}{\begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}}$$

$$I_1'' = \frac{0 - Z_3 E_2}{\left((Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2\right)}$$

$$\boxed{I_1'' = \frac{-Z_3 E_2}{(Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1)}} \qquad (11)$$

And,

$$I_2'' = \frac{\begin{bmatrix} Z_1+Z_3 & 0 \\ Z_3 & E_2 \end{bmatrix}}{\begin{bmatrix} Z_1+Z_3 & Z_3 \\ Z_3 & Z_2+Z_3 \end{bmatrix}}$$

$$I_2'' = \frac{(Z_1+Z_3)E_2}{\left((Z_1+Z_3)(Z_2+Z_3)-Z_3^2\right)}$$

$$\boxed{I_2'' = \frac{(Z_1+Z_3)E_2}{(Z_1Z_2+Z_2Z_3+Z_3Z_1)}} \qquad (12)$$

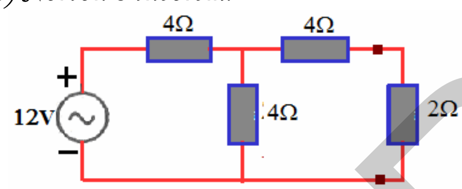
Putting values of I_1', I_2', I_1'' *and* I_2'' *in eqs. (3) and (4) we have,*

$$\boxed{I_1 = I_1' + I_1''}$$

$$\boxed{I_2 = I_2' + I_2''}$$

Thus the theorem is proved.

Example 4: Find the current in 2Ω of impedance in the following circuit using (a) Thevenin's theorem and (b) Norton's theorem.



Solution:

(a) From the given circuit we can write,

$Z_1=4\,\Omega$, $Z_2=4\,\Omega$, $Z_3=4\,\Omega$, $Z_L=2\,\Omega$ and $E=12V$

$$E' = \frac{Z_3}{(Z_1+Z3)}E = \frac{4}{4+4}\times 12 = \frac{4}{8}\times 12 = 6V$$

$$Z' = Z_2 + \frac{Z_1Z3}{(Z_1+Z3)} = 4 + \frac{4\times 4}{4+4} = 4+2 = 6\Omega$$

$$I_L = \frac{E'}{Z'+Z_L} = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4} = 0.75amp$$

(b) From the given circuit we can write,

$Z_1=4\,\Omega$, $Z_2=4\,\Omega$, $Z_3=4\,\Omega$, $Z_L=2\,\Omega$ and $E=12V$

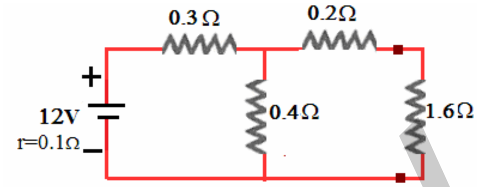
$$I' = \frac{Z_3E}{(Z_1Z_2+Z_2Z_3+Z_3Z_1)}$$

$$I' == \frac{4\times 12}{4\times 4+4\times 4+4\times 4} = \frac{48}{48} = 1amp$$

$$Z' = Z_2 + \frac{Z_1Z3}{(Z_1+Z3)} = 4 + \frac{4\times 4}{4+4} = 4+2 = 6\Omega$$

$$I_L = \frac{Z'I'}{Z'+Z_L} = \frac{6\times 1}{6+2} = \frac{6}{8} = \frac{3}{4} = 0.75amp$$

Example 5: Find the current in 1.6Ω of resistance and draw the equivalent circuit for the following network using (a) Thevenin's theorem and (b) Norton's theorem.



(a) From the given circuit we can write,

$Z_1=0.3+0.1=0.4\,\Omega$, $Z_2=0.2\,\Omega$,

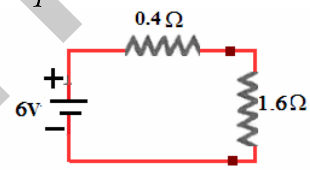
$Z_3=0.4\,\Omega$, $Z_L=1.6\,\Omega$ *and* $E=12V$

$$E' = \frac{Z_3}{(Z_1+Z3)}E = \frac{0.4}{0.4+0.4}\times 12 = \frac{4}{8}\times 12 = 6V$$

$$Z' = Z_2 + \frac{Z_1Z3}{(Z_1+Z3)}$$

$$Z' = 0.2 + \frac{0.4\times 0.4}{0.4+0.4} = 0.2+0.2 = 0.4\Omega$$

Thus voltage equivalent circuit can be drawn as,



Hence the load current ,

$$I_L = \frac{E'}{Z'+Z_L} = \frac{6}{1.6+0.4} = \frac{6}{2} = 3amp$$

(b) From the given circuit we can write,

$Z_1=0.3+0.1=0.4\,\Omega$, $Z_2=0.2\,\Omega$,

$Z_3=0.4\,\Omega$, $Z_L=1.6\,\Omega$ *and* $E=12V$

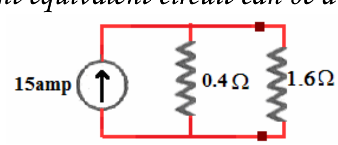
$$I' = \frac{Z_3E}{(Z_1Z_2+Z_2Z_3+Z_3Z_1)}$$

$$I' = \frac{0.4\times 12}{0.4\times 0.2+0.2\times 0.4+0.4\times 0.4}$$

$$I' = \frac{4.8}{0.08+0.08+.16} = \frac{4.8}{0.32} = \frac{30}{2} = 15amp$$

$$Z' = 0.2 + \frac{0.4\times 0.4}{0.4+0.4} = 0.2+0.2 = 0.4\Omega$$

Thus current equivalent circuit can be drawn as,,



Hence the load current ,

$$I_L = \frac{Z'I'}{Z'+Z_L} = \frac{0.4\times 15}{0.4+1.6} = \frac{6}{2} = 3amp$$