Lecture 4&5: LCR circuit

Charging of capacitor through inductor and resistor

Let us consider a capacitor of capacitance C is connected to a DC source of e.m.f. E through a resister of resistance R , an inductor of inductance L and a key K in series. When the key K is switched on, the charging process of capacitor starts instantaneous current. The charge on capacitor increases with time and attains its maximum in certain duration of time. As the time passes, the charge on capacitor increases and gets its maximum. The nature of charge increase depends on the value of inductor and resistor. The nature of charge increase can be analyzed in the following ways.

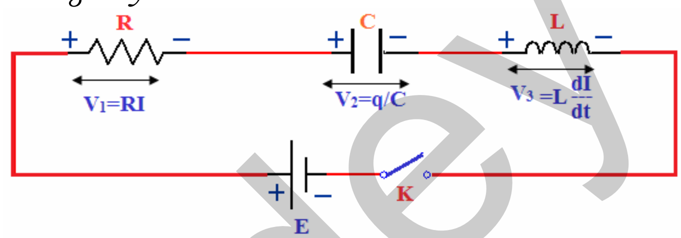

Fig.1

According to KVL, the algebraic sum of instantaneous voltage drop across the circuit elements for a closed loop is zero.

Thus $E - V_1 - V_2 - V_3 = 0$

$$E = V_1 + V_2 + V_3 \qquad (1)$$

If the current in circuit at time t is 'i' and charge stored on capacitor is q then the potential drop across resistor, capacitor and inductor will be Ri, q/C and $L\frac{di}{dt}$ *respectively.*

Using eq.(1)

$$E = Ri + \frac{q}{C} + L\frac{di}{dt}$$

$$\Rightarrow \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} + \frac{d^2q}{dt^2} = \frac{E}{L}$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{E}{L} \qquad (2)$$

Since dimension of 1/LC is T^{-1} *and dimension of L/R is* T. *Thus we can consider that:*

$$\omega^2 = \frac{1}{LC} \text{ and } 2k = \frac{1}{\tau} = \frac{R}{L}$$

Here ω *is angular frequency of oscillation. And* τ *is time constant. The constant k is called as damping constant. Now eq.(2) becomes as;*

$$\Rightarrow \frac{d^2q}{dt^2} + 2k\frac{dq}{dt} + \omega^2 q = \frac{E}{L}$$

$$\frac{d^2q}{dt^2} + 2k\frac{dq}{dt} + \omega^2\left(q - \frac{E}{\omega^2 L}\right) = 0 \qquad (3)$$

Since, $\frac{E}{\omega^2 L} = \frac{LCE}{L} = CE = q_0$ *, thus eq.(3) becomes as,*

$$\frac{d^2q}{dt^2} + 2k\frac{dq}{dt} + \omega^2(q - q_0) = 0$$

Let, $x = q - q_0$ *then*

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0 \qquad (4)$$

The eq.(4) is differential equation for growth charge across the plates of capacitor in LCR circuit. Consider the solution of eq.(4) is:

$$x = A e^{\alpha t} \qquad (5)$$

Differentiating eq.(5) w.r.t. time t we have:

$$\frac{dx}{dt} = \alpha A e^{\alpha t} = \alpha x$$

$$\frac{d^2x}{dt^2} = \alpha^2 A e^{\alpha t} = \alpha^2 x$$

Putting value of $\frac{dx}{dt}$ *and* $\frac{d^2x}{dt^2}$ *in eq.(4) we get,*

$$\alpha^2 x + 2k\,\alpha x + \omega^2 x = 0$$

$$\alpha^2 + 2k\,\alpha + \omega^2 = 0 \qquad (6)$$

Eq. (6) provides that,

$$\alpha = \frac{-2k \pm \sqrt{4k^2 - 4\omega^2}}{2} = -k \pm \sqrt{k^2 - \omega^2}$$

Thus constant α *has two values, say they are* α_1 *and* α_2 *then.*

$$\alpha_1 = -k + \sqrt{k^2 - \omega^2}$$

and $\alpha_2 = -k - \sqrt{k^2 - \omega^2}$

Since α *has two values thus solution of eq.(4) can be written as linear combination of* $e^{\alpha_1 t}$ and $e^{\alpha_2 t}$.

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

$$q - q_0 = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

$$q = q_0 + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \qquad (7)$$

Differentiating eq.(7) w.r.to time t we get:

$$I = \frac{dq}{dt} = A_1 \alpha_1 e^{\alpha_1 t} + A_2 \alpha_2 e^{\alpha_2 t} \qquad (8)$$

The constants A_1 *and* A_2 *can be determined by applying initial condition.*

The initial conditions are:

At t=0, q=0 and $I = \frac{dq}{dt} = 0$

Under these conditions the eqs.(7) and (8) provides that,

$$0 = q_0 + A_1 + A_2$$

$$A_1 + A_2 = -q_0 \qquad (9)$$

And

$$0 = A_1 \alpha_1 + A_2 \alpha_2$$

$$A_1\left(-k + \sqrt{k^2 - \omega^2}\right) + A_2\left(-k - \sqrt{k^2 - \omega^2}\right) = 0$$

$$-k(A_1 + A_2) + \sqrt{k^2 - \omega^2}\,(A_1 - A_2) = 0$$

$$k q_0 + \sqrt{k^2 - \omega^2}\,(A_1 - A_2) = 0$$

$$(A_1 - A_2) = -\frac{k q_0}{\sqrt{k^2 - \omega^2}} \qquad (10)$$

Solving eqs. (9) and (10), we have:

$$A_1 = -\frac{q_0}{2}\left(1 + \frac{k}{\sqrt{k^2 - \omega^2}}\right)$$

$$A_2 = -\frac{q_0}{2}\left(1 - \frac{k}{\sqrt{k^2 - \omega^2}}\right)$$

Putting values A_1, A_2, α_1 *and* α_2 *in eq.(7), we have:*

$$q = q_0 - \frac{q_0}{2}\left(1 + \frac{k}{\sqrt{k^2 - \omega^2}}\right) e^{\left(-k + \sqrt{k^2 - \omega^2}\right)t} - \frac{q_0}{2}\left(1 - \frac{k}{\sqrt{k^2 - \omega^2}}\right) e^{\left(-k - \sqrt{k^2 - \omega^2}\right)t} \qquad (11)$$

The eq.(11) is the general expression for growth of charge across the plates of capacitor in LCR circuit. This equation can be analyzed in the terms of following three cases.

Case1: Heavily damped or dead beat or over damped condition:

The condition, in which the damping term is greater than the oscillatory term, is called as over damped condition.

i.e. $k^2 > \omega^2$ or $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$ or $R > 2\sqrt{\frac{L}{C}}$

In this condition, $\sqrt{k^2 - \omega^2}$ = *positive* $< k$

Hence, $-k \pm \sqrt{k^2 - \omega^2}$ = negative

Therefore the second and third RHS terms of eq.(11) decay exponentially and becomes equal to zero at t=∞. As a result q approaches to q_0. *i.e. In the dead beat condition, the charge on capacitor increases with time and after t=∞ it gains its maximum value (Fig.2).*

Case2: Critically damped condition:

The condition, in which the damping term is approximately equal to the oscillatory term, is called as critically damped condition.

i.e. $k^2 \approx \omega^2$ or $\left(\frac{R}{2L}\right)^2 \approx \frac{1}{LC}$ or $R \approx 2\sqrt{\frac{L}{C}}$

In this condition, $\sqrt{k^2 - \omega^2}$ = *positive* $= h$

Hence, $-k \pm \sqrt{k^2 - \omega^2} = -k \pm h$

So, The eq.(7) becomes as,

$$q = q_0 + A_1 e^{(-k+h)t} + A_2 e^{(-k-h)t}$$

$$q = q_0 + e^{-kt}\left\{A_1 e^{ht} + A_2 e^{-ht}\right\}$$

$$q = q_0 + e^{-kt}\{A_1(1+ht) + A_2(1-ht)\}$$

$$q = q_0 + e^{-kt}\{(A_1 + A_2) + (A_1 - A_2)ht\}$$

$$q = q_0 + e^{-kt}\left\{-q_0 - \frac{kq_0}{\sqrt{k^2-\omega^2}}ht\right\}$$

$$q = q_0 + e^{-kt}\left\{-q_0 - \frac{kq_0}{h}ht\right\}$$

$$\boxed{q = q_0 - q_0\{1+kt\}e^{-kt}} \qquad (12)$$

The second term of eq.(12) show the fast decay of charge and becomes equal to zero in a small time. As a result, the charge on Capacitor increases rapidly with time and it gains its maximum value in very soon time (Fig.2).

Case3: Lightly damped condition:

The condition, in which the damping term is less than the oscillatory term, is called as lightly damped or damped harmonic condition.

$$\text{i.e. } k^2 < \omega^2 \text{ or } \left(\frac{R}{2L}\right)^2 < \frac{1}{LC} \text{ or } R < 2\sqrt{\frac{L}{C}}$$

$$\text{Thus, } \sqrt{k^2-\omega^2} = \sqrt{-(\omega^2-k^2)} = j\beta$$

$$\text{Hence, } -k \pm \sqrt{k^2-\omega^2} = -k \pm j\beta$$

So, The eq.(11) becomes as,

$$q = q_0 - \frac{q_0}{2}\left(1+\frac{k}{j\beta}\right)e^{(-k+j\beta)t} - \frac{q_0}{2}\left(1-\frac{k}{j\beta}\right)e^{(-k-j\beta)t}$$

$$q = q_0 - q_0e^{-kt}\left\{\frac{1}{2}\left(1+\frac{k}{j\beta}\right)e^{j\beta t} + \frac{1}{2}\left(1-\frac{k}{j\beta}\right)e^{-j\beta t}\right\}$$

$$q = q_0 - q_0e^{-kt}\left\{\frac{e^{j\beta t}+e^{-j\beta t}}{2} + \frac{k}{\beta}\frac{e^{j\beta t}-e^{-j\beta t}}{2j}\right\}$$

$$q = q_0 - q_0e^{-kt}\left\{\cos\beta t + \frac{k}{\beta}\sin\beta t\right\} \qquad (13)$$

$$\text{Let } \frac{k}{\beta} = \tan\phi = \frac{\sin\phi}{\cos\phi}$$

$$\text{Then } \cos\phi = \frac{\beta}{\sqrt{k^2+\beta^2}} = \frac{\beta}{\omega}$$

Hence eq.(13) becomes as,

$$q = q_0 - q_0e^{-kt}\left\{\cos\beta t + \frac{\sin\phi}{\cos\phi}\sin\beta t\right\}$$

$$q = q_0 - \frac{q_0e^{-kt}}{\cos\phi}\{\cos\beta t\ \cos\phi + \sin\beta t\ \sin\phi\}$$

$$q = q_0 - \frac{q_0\omega e^{-kt}}{\beta}\cos(\beta t - \phi)$$

$$\boxed{q = q_0 - \frac{q_0\omega}{\beta}\cos(\beta t - \phi)\ e^{-kt}} \qquad (14)$$

The second term of eq.(14) show exponential harmonic decay. Thus the charge on capacitor increases with time and it oscillates about q_0 whose amplitude decreases exponentially. After t=∞ it saturates to its maximum value (Fig.2).

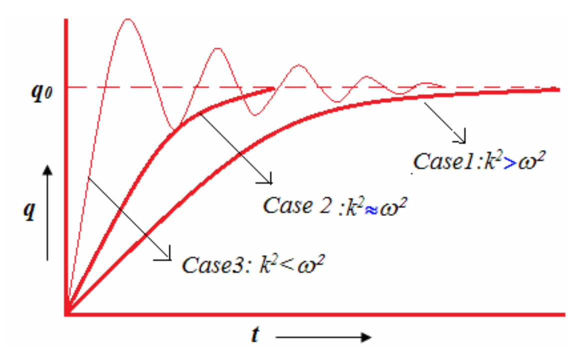

Fig.2

Lecture 4: LCR circuit

Discharging of capacitor through inductor and resistor

Let us consider a charged capacitor of capacitance C is connected to a resister of resistance R, an inductor of inductance L and a key K in series. When the key K is switched on, the discharging process of capacitor starts through inductor and resistor. The charge on capacitor decreases with time due to loss of energy through resistor. The nature of charge decrease depends on the value of inductor and resistor which can be analyzed in the following ways.

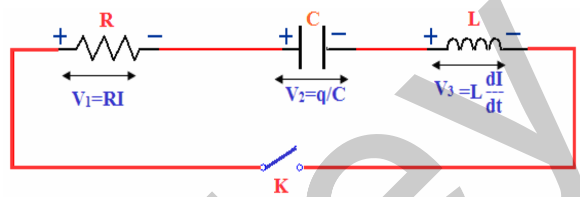

Fig.1

According to KVL, the algebraic sum of instantaneous voltage drop across the circuit elements for a closed loop is zero.

Thus $E - V_1 - V_2 - V_3 = 0$

$$E = V_1 + V_2 + V_3$$

As E=0, $V_1 + V_2 + V_3 = 0$ *(1)*

If the current in circuit at time t is 'i' and charge remained on capacitor is q then the potential drop across resistor, capacitor and inductor will be Ri, q/C and $L\dfrac{di}{dt}$ *respectively.*

Using eq.(1)

$$Ri + \frac{q}{C} + L\frac{di}{dt} = 0$$

$$\Rightarrow \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} + \frac{d^2q}{dt^2} = 0$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 \qquad (2)$$

Since dimension of 1/LC is T^{-1} *and dimension of L/R is* T. *Thus we can consider that:*

$$\omega^2 = \frac{1}{LC} \text{ and } 2k = \frac{1}{\tau} = \frac{R}{L}$$

here ω *is angular frequency of oscillation. And* τ *is time constant. The constant k is called as damping constant.*

Now eq.(2) becomes as;

$$\Rightarrow \frac{d^2q}{dt^2} + 2k\frac{dq}{dt} + \omega^2 q = 0 \qquad (3)$$

The eq.(3) is differential equation for decay of charge across the plates of capacitor in LCR circuit. Consider the solution of eq.(3) is:

$$q = A e^{\alpha t} \qquad (4)$$

Differentiating eq.(4) w.r.t. time t we have:

$$\frac{dq}{dt} = \alpha A e^{\alpha t} = \alpha q$$

$$\frac{d^2q}{dt^2} = \alpha^2 A e^{\alpha t} = \alpha^2 q$$

Putting value of $\dfrac{dq}{dt}$ *and* $\dfrac{d^2q}{dt^2}$ *in eq.(3) we get,*

$$\alpha^2 q + 2k\alpha q + \omega^2 q = 0$$

$$\alpha^2 + 2k\alpha + \omega^2 = 0 \qquad (5)$$

Eq. (5) provides that,

$$\alpha = \frac{-2k \pm \sqrt{4k^2 - 4\omega^2}}{2} = -k \pm \sqrt{k^2 - \omega^2}$$

Thus constant α *has two values, say they are* α_1 *and* α_2 *then.*

$$\alpha_1 = -k + \sqrt{k^2 - \omega^2}$$

and $$\alpha_2 = -k - \sqrt{k^2 - \omega^2}$$

Since α *has two values thus solution of eq.(3) Can be written as linear combination of* $e^{\alpha_1 t}$ and $e^{\alpha_2 t}$.

$$q = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \qquad (6)$$

Differentiating eq.(6) w.r.to time t we get:

$$I = \frac{dq}{dt} = A_1 \alpha_1 e^{\alpha_1 t} + A_2 \alpha_2 e^{\alpha_2 t} \qquad (7)$$

The constants A_1 *and* A_2 *can be determined by applying initial condition.*

The initial conditions are:

At t=0, $q = q_0$ *and* $I = \frac{dq}{dt} = 0$

Under these conditions the eqs.(6) and (7) provides that,

$$q_0 = A_1 + A_2$$

$$A_1 + A_2 = q_0 \qquad (8)$$

And

$$0 = A_1 \alpha_1 + A_2 \alpha_2$$

$$A_1\left(-k + \sqrt{k^2 - \omega^2}\right) + A_2\left(-k - \sqrt{k^2 - \omega^2}\right) = 0$$

$$-k(A_1 + A_2) + \sqrt{k^2 - \omega^2}\,(A_1 - A_2) = 0$$

$$-k\,q_0 + \sqrt{k^2 - \omega^2}\,(A_1 - A_2) = 0$$

$$(A_1 - A_2) = \frac{k\,q_0}{\sqrt{k^2 - \omega^2}} \qquad (9)$$

Solving eqs. (8) and (9), we have:

$$A_1 = \frac{q_0}{2}\left(1 + \frac{k}{\sqrt{k^2 - \omega^2}}\right)$$

$$A_2 = \frac{q_0}{2}\left(1 - \frac{k}{\sqrt{k^2 - \omega^2}}\right)$$

Putting values A_1, A_2, α_1 *and* α_2 *in eq.(6), we have:*

$$q = \frac{q_0}{2}\left(1 + \frac{k}{\sqrt{k^2 - \omega^2}}\right)e^{\left(-k + \sqrt{k^2 - \omega^2}\right)t} + \frac{q_0}{2}\left(1 - \frac{k}{\sqrt{k^2 - \omega^2}}\right)e^{\left(-k - \sqrt{k^2 - \omega^2}\right)t} \qquad (10)$$

The eq.(10) is the general expression for decay of charge across the plates of capacitor in LCR circuit. This equation can be analyzed in the terms of following three cases.

Case1: Heavily damped or dead beat or over damped condition:

The condition, in which the damping term is greater than the oscillatory term, is called as over damped condition.

i.e. $k^2 > \omega^2$ or $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$ or $R > 2\sqrt{\frac{L}{C}}$

In this condition, $\sqrt{k^2 - \omega^2}$ = *positive* $< k$

Hence, $-k \pm \sqrt{k^2 - \omega^2}$ = negative

Therefore both the RHS terms of eq.(10) decay exponentially and tends to zero at t=∞. Thus, in the dead beat condition, the charge on capacitor decreases exponentially with time and after t=∞ it tends to zero. (Fig.2).

Case2: Critically damped condition:

The condition, in which the damping term is approximately equal to the oscillatory term, is called as critically damped condition.

i.e. $k^2 \approx \omega^2$ or $\left(\frac{R}{2L}\right)^2 \approx \frac{1}{LC}$ or $R \approx 2\sqrt{\frac{L}{C}}$

In this condition, $\sqrt{k^2 - \omega^2}$ = *positive* $= h$

Hence, $-k \pm \sqrt{k^2 - \omega^2} = -k \pm h$

So, The eq.(6) becomes as,

$$q = A_1 e^{(-k+h)t} + A_2 e^{(-k-h)t}$$

$$q = e^{-kt}\left\{A_1 e^{ht} + A_2 e^{-ht}\right\}$$

$$q = e^{-kt}\{A_1(1 + ht) + A_2(1 - ht)\}$$

$$q = e^{-kt}\{(A_1 + A_2) + (A_1 - A_2)ht\}$$

$$q = e^{-kt}\left\{q_0 + \frac{kq_0}{\sqrt{k^2 - \omega^2}}ht\right\}$$

$$q = e^{-kt}\left\{q_0 + \frac{kq_0}{h}ht\right\}$$

$$q = q_0\{1 + kt\}e^{-kt} \qquad (11)$$

The eq.(11) show that the charge on capacitor decreases very fast with time due to term te^{-kt} *and it tends to zero in very soon time (Fig.2).*

Case3: Lightly damped condition:

The condition, in which the damping term is less than the oscillatory term, is called as lightly damped or damped harmonic condition.

i.e. $k^2 < \omega^2$ or $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$ or $R < 2\sqrt{\frac{L}{C}}$

Thus, $\sqrt{k^2-\omega^2} = \sqrt{-(\omega^2-k^2)} = j\beta$

Hence, $-k \pm \sqrt{k^2-\omega^2} = -k \pm j\beta$

So, The eq.(11) becomes as,

$$q = \frac{q_0}{2}\left(1+\frac{k}{j\beta}\right)e^{(-k+j\beta)t} + \frac{q_0}{2}\left(1-\frac{k}{j\beta}\right)e^{(-k-j\beta)t}$$

$$q = q_0 e^{-kt}\left\{\frac{1}{2}\left(1+\frac{k}{j\beta}\right)e^{j\beta t} + \frac{1}{2}\left(1-\frac{k}{j\beta}\right)e^{-j\beta t}\right\}$$

$$q = q_0 e^{-kt}\left\{\frac{e^{j\beta t}+e^{-j\beta t}}{2} + \frac{k}{\beta}\,\frac{e^{j\beta t}-e^{-j\beta t}}{2j}\right\}$$

$$q = q_0 e^{-kt}\left\{\cos\beta t + \frac{k}{\beta}\sin\beta t\right\} \quad (12)$$

Let $\frac{k}{\beta} = \tan\phi = \frac{\sin\phi}{\cos\phi}$

Then $\cos\phi = \frac{\beta}{\sqrt{k^2+\beta^2}} = \frac{\beta}{\omega}$

Hence eq.(12) becomes as,

$$q = q_0 e^{-kt}\left\{\cos\beta t + \frac{\sin\phi}{\cos\phi}\sin\beta t\right\}$$

$$q = \frac{q_0 e^{-kt}}{\cos\phi}\{\cos\beta t\ \cos\phi + \sin\beta t\ \sin\phi\}$$

$$q = \frac{q_0\omega}{\beta}\cos(\beta t - \phi)\, e^{-kt} \quad (13)$$

The eq.(13) shows that the There is exponential harmonic decay of charge on capacitor with time. i.e. Charge on capacitor oscillates about q=0 whose amplitude decays exponentially. After t=∞ it tends to zero (Fig.2). Thus, there occurs a damped harmonic oscillation of charge on capacitor in case of light damping.

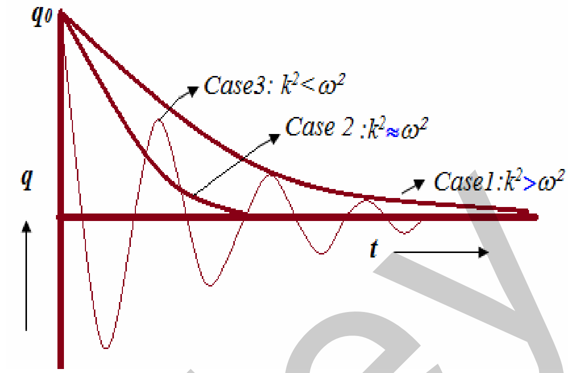

Fig.2

NoteA: The quality factor for LCR circuit under lightly damped condition is $\omega\tau = \omega L/R$.

NoteB: If f is the frequency of oscillation of charge under lightly damped condition then,

Since $\beta = \sqrt{\omega^2-k^2}$

$$2\pi f = \sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

Que: A capacitor of 2μf, an inductor of 0.8mH and a resistor of 20Ω are connected in series. Is the circuit is oscillatory? If yes, then calculate its frequency.

Ans: Given that, C=2μf, L=0.8mH, R=20 Ω

$$2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{0.8\times10^{-3}}{2\times10^{-6}}} = 2\sqrt{4\times10^2} = 40\ \Omega$$

Since $R < 2\sqrt{\frac{L}{C}}$ thus the circuit is oscillatory.

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{0.8\times10^{-3}\times2\times10^{-6}} - \frac{(20)^2}{4\times(0.8\times10^{-3})^2}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{10^{10}}{16}-\frac{10^{10}}{64}} = \frac{10^5}{2\pi}\sqrt{\frac{3}{64}} = \frac{10^5}{2\times3.14}\times\frac{1.732}{8}$$

$$f = \frac{1.732\times10^5}{50.24} = 0.0345\times10^5\,Hz = 3.45\times10^3\,Hz$$

$$f = 3.45kHz$$