Lecture 3: LC circuit

Discharging of capacitor through inductor

Let us consider a charged capacitor of capacitance C is connected to a pure inductor of inductance L through a key K in series. When the key K is switched on, the capacitor starts discharging through inductor. After certain time capacitor is fully discharged and inductor becomes fully charged. The potential difference across the inductor becomes equal and opposite to that of capacitor due to no energy loss. That capacitive energy ($q_0^2/2C$) is converted to inductive energy ($Li_0^2/2$). Now, inductor starts to charge the capacitor due to high potential difference across it. In this process, the inductive energy converts in to capacitive energy. Due to no loss of energy, this transfer of energy L→C and C→L is continues for the infinite time.

Thus in discharging process of a capacitor through inductor, there is an oscillation of energy/voltage/charge from capacitor to inductor and inductor to capacitor.

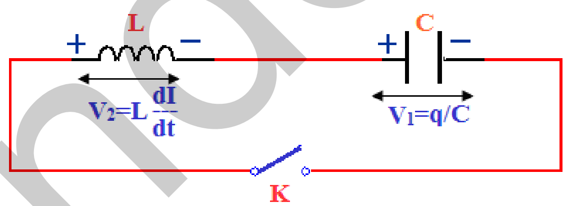

Fig.1

According to KVL, the algebraic sum of instantaneous voltage drop across the circuit elements for a closed loop is zero.

Thus $E - V_1 - V_2 = 0$

$E = V_1 + V_2$

Here E=0, Thus $V_1 + V_2 = 0$ *(1)*

If the current in circuit at time t is I and charge stored on capacitor is q then the potential drop across inductor and capacitor will be $L\frac{di}{dt}$ *and q/C respectively.*

Using eq.(1)

$$\frac{q}{C} + L\frac{di}{dt} = 0$$

$$\Rightarrow \frac{q}{LC} + \frac{d^2q}{dt^2} = 0$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \qquad (2a)$$

Since dimension of 1/LC is T^{-1} *thus consider that* $\omega^2 = \frac{1}{LC}$; ω *is angular frequency of oscillation. Hence eq.(2b) becomes as;*

$$\Rightarrow \frac{d^2q}{dt^2} + \omega^2 q = 0 \qquad (2b)$$

Eq.(2b) is similar to differential equation of S.H.M. Hence the oscillation of charge follows the simple harmonic oscillation.

Let the solution of eq.(2b) is:

$$q = A\,e^{\alpha t} \qquad (3)$$

Differentiating two times eq.(3) w.r.to time t we have:

$$\frac{d^2q}{dt^2} = \alpha^2 A\,e^{\alpha t} = \alpha^2 q \qquad (4)$$

From eqs. (2b) and (4)

$$\alpha^2 q + \omega^2 q = 0$$

$$\alpha^2 + \omega^2 = 0$$

$$\alpha^2 = -\omega^2$$

$$\alpha = \pm j\omega$$

Since α *has two values thus the total solution can be written as linear combination* $e^{j\omega t}$ *and* $e^{-j\omega t}$. *Therefore eq.(3) becomes as:*

$$q = A_1\,e^{j\omega t} + A_2\,e^{-j\omega t} \qquad (5)$$

Differentiating eq.(5) w.r.to time t we get:

$$I = \frac{dq}{dt} = A_1\,j\omega\,e^{j\omega t} - A_2\,j\omega\,e^{-j\omega t} \qquad (6)$$

The constants A_1 and A_2 can be determined by applying initial condition. The initial conditions are:

At t=0, $q=q_0$ *and* $I=\frac{dq}{dt}=0$

Under these conditions the eqs.(5) and (6) becomes as,

$$q_0 = A_1 + A_2$$

$$A_1 + A_2 = q_0 \qquad (7)$$

And $0 = A_1\, j\omega - A_2\, j\omega$

$$A_1 - A_2 = 0 \qquad (8)$$

Solving eqs. (7) and (8), we have:

$$A_1 = A_2 = \frac{q_0}{2} \qquad (9)$$

Putting values from eq.(9) to eq.(6), we have:

$$q = \frac{q_0}{2} e^{j\omega t} + \frac{q_0}{2} e^{-j\omega t}$$

$$q = q_0\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)$$

$$q = q_0 \cos(\omega t) \qquad (10)$$

Thus in discharging process of capacitor trough inductor, the charge on capacitor is oscillatory in nature (fig.2). If T and f are the time period and frequency of oscillation then,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

And $f = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$

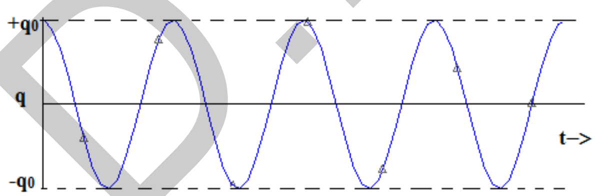

Fig.2

Note 1: Oscillation in LC circuit is equivalent to oscillation in spring-mass system. In this case, the energy alternates between kinetic to potential. The current in the LC-circuit is zero when the entire energy is stored in the capacitor and it is maximum when the entire energy is stored as magnetic energy in the inductor. In the same way the velocity of mass is zero when the entire energy is potential energy in spring, and is maximum when the total energy is kinetic energy of mass. Thus capacitor is like a spring while inductor is like a mass. i.e.

$$\frac{1}{2} L i^2 \equiv \frac{1}{2} m v^2 \Rightarrow (L \equiv m)$$

$$\frac{1}{2}\frac{q^2}{C} \equiv \frac{1}{2} k x^2 \Rightarrow (C \equiv \frac{1}{k})$$

Example 1: A 20/π μf of a capacitor is discharged through a 50/π mH an inductor. Calculate the frequency of discharge.

Solution: Given that,

C= 20/π μf, L=50/π mH, f=?

We know that

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{\frac{50}{\pi}\times 10^{-3} \times \frac{20}{\pi}\times 10^{-6}}}$$

$$f = \frac{1}{2\sqrt{10^{-6}}} = \frac{1}{2\times 10^{-3}} = \frac{1000}{2}$$

$$f = 500 Hz$$

Example 2: A 20 μf of a capacitor is discharging through a 50mH an inductor. If the maximum amplitude of an oscillatory charge in LC circuit is 0.4μC then find the maximum amplitude of current?

Solution: Given that, C= 20 μf, L=50 mH, q_0*=0.4μC,* i_0*=?*

We know that

$$q = q_0 \cos(\omega t)$$

Differentiating this equation

$$i = \frac{dq}{dt} = -q_0 \omega \sin(\omega t)$$

Thus $i_0 = q_0\omega = \frac{q_0}{\sqrt{LC}}$

$$i_0 = \frac{0.4\times 10^{-6}}{\sqrt{50\times 10^{-3} \times 20\times 10^{-6}}}$$

$$i_0 = 0.4\times 10^{-3} amp = 0.4 mA$$

