Electrostatic Potential and potential difference: *It is work done to bring a unit charge from infinite in an electric field. It is denoted by ϕ and is measured in Joule/coulomb.*

$$\phi = \frac{W_{\infty\to r}}{q_0} = -\int_{\infty}^{r} \frac{\vec{F}.d\vec{l}}{q_0} = -\int_{\infty}^{r} \frac{q_0\vec{E}.d\vec{l}}{q_0} = -\int_{\infty}^{r} \vec{E}.d\vec{l} \qquad (1)$$

Let ϕ_1 and ϕ_2 are the electrostatic potential at points A and B which are at distance r_1 and r_2 respectively form the source of field. Then from equation (1), we can write,

$$\phi_1 = -\int_{\infty}^{r_1} \vec{E}.d\vec{l} \quad and \quad \phi_2 = -\int_{\infty}^{r_2} \vec{E}.d\vec{l}$$

Hence,

$$\phi_2 - \phi_1 = -\int_{\infty}^{r_2} \vec{E}.d\vec{l} + \int_{\infty}^{r_1} \vec{E}.d\vec{l} = -\left[\int_{\infty}^{r_2} \vec{E}.d\vec{l} - \int_{\infty}^{r_1} \vec{E}.d\vec{l}\right] = -\left[\int_{r_1}^{\infty} \vec{E}.d\vec{l} + \int_{\infty}^{r_2} \vec{E}.d\vec{l}\right] = -\int_{r_1}^{r_2} \vec{E}.d\vec{l}$$

$$\phi_2 - \phi_1 = -\int_{r_1}^{r_2} \vec{E}.d\vec{l} \qquad (2)$$

Thus line integral of electric field over any path between two points gives the potential difference between them. The equation (2) can also be written as,

$$\int_{r_1}^{r_2} \left(\vec{\nabla}\phi.d\vec{l}\right) = -\int_{r_1}^{r_2} \vec{E}.d\vec{l}$$

$$\Rightarrow \quad \boxed{\vec{E} = -\vec{\nabla}\phi}$$

This shows that if the electric field is irrotational and conservative field. i.e. $\vec{\nabla}\times\vec{E} = 0$ or $\oint \vec{E}\cdot d\vec{l} = 0 \;\; \Rightarrow \int \vec{E}\cdot d\vec{l}$ is path independent then it can be written as negative gradient of of a scalar function which is termed as electrostatic potential.

Alternate method to prove $\vec{E} = -\vec{\nabla}\phi$

Let ϕ and $\phi + d\phi$ are the electric potential at two close points having co-ordinates (x,y,z) and (x+dx, y+dy, z+dz) respectively. The small distance between two points can be written as,

$$d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Since ϕ is function of position co-ordinates thus change in ϕ corresponding to small displacement $d\vec{l}$ can be written as,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$d\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right).\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$d\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi.\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$d\phi = \vec{\nabla}\,\phi.\,d\vec{l} \qquad (2)$$

Since line integral of electric field over a small distance provides potential difference between two points thus,

$$d\phi = -\vec{E}.\,d\vec{l} \qquad (3)$$

Comparing equation (2) and (3) we have,

$$\boxed{\vec{E} = -\vec{\nabla}\phi}$$

$$\Rightarrow \quad E_x = -\frac{\partial\phi}{\partial x}, \; E_y = -\frac{\partial\phi}{\partial y} \; and \; E_z = -\frac{\partial\phi}{\partial z}$$

D K Pandey

Electrostatic Potential due to different electrostatic system:

(1) *Due to Point charge:* The electric field due to point charge q at distance r is given by following formula.

$$\vec{E} = \frac{1}{4\pi\varepsilon}\frac{q}{r^3}\vec{r}$$

Since $\phi = -\int_{\infty}^{r} \vec{E}.d\vec{l}$ and $d\vec{l} = d\vec{r}$

Thus $$\phi = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon}\frac{q}{r^3}\vec{r}.d\vec{r} = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{r}\frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{\infty}\right] = \frac{1}{4\pi\varepsilon}\frac{q}{r}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\frac{q}{r}}$$

(2) *Due to distribution of Point charge:* Suppose q_1, q_2, q_3, ….. charges are distributed in medium having ε permitivity . These charges at distances r_1, r_2, r_3, …. from point a point P. If the potential at this point is ϕ then it will be scalar sum of all potentials of individual charges.

$$\phi = \phi_1 + \phi_2 + \phi_3 + \cdots$$

$$\phi = \frac{1}{4\pi\varepsilon}\frac{q_1}{r_1} + \frac{1}{4\pi\varepsilon}\frac{q_2}{r_2} + \frac{1}{4\pi\varepsilon}\frac{q_3}{r_3} + \cdots$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\sum_i \frac{q_i}{r_i}}$$

(3) *Due to uniform charge distribution:* Suppose dq charges are uniformly distributed over a region. If ϕ is potential at distance r from this charge distribution then,

$$\phi = \frac{1}{4\pi\varepsilon}\int\frac{dq}{r}$$

For line charge distribution, $\phi = \frac{1}{4\pi\varepsilon}\int\frac{\lambda dl}{r}$

For surface charge distribution, $\phi = \frac{1}{4\pi\varepsilon}\int\frac{\sigma ds}{r}$

For volume charge distribution, $\phi = \frac{1}{4\pi\varepsilon}\int\frac{\rho dv}{r}$

Electrostatic Potential due to uniformly charge spherical shell :

Suppose q charge is uniformly distributed over a spherical shell of radius of R. If σ is the surface charge density of distribution then $q = 4\pi R^2\sigma$. The electric field at external and internal point due to this charge distribution is given by following expressions.

$$\vec{E}_{ext} = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r} \text{ and } \vec{E}_{int} = 0$$

The electrostatic potential due to this electrostatic system at external, surface and internal point can be obtained in following manner.

(1) At external point: The electric potential can be found by line integral of electric field over range ∞ to r (>R). i.e.

$$\because \quad \phi = -\int_{\infty}^{r}\vec{E}.d\vec{r} \qquad \therefore \quad \phi = -\int_{\infty}^{r}\frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}.d\vec{r} = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{r}\frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{\infty}^{r} = \frac{1}{4\pi\varepsilon}\frac{q}{r}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\frac{q}{r} = \frac{\sigma}{\varepsilon}\frac{R^2}{r}}$$

D K Pandey

Do not publish it. Copy righted material.

(1) At surface point: The electric potential can be found by line integral of electric field over range ∞ to R, i.e.

$$\phi = -\int_{\infty}^{R} \vec{E}.d\vec{r}$$

$$\therefore \quad \phi = -\int_{\infty}^{R} \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}.d\vec{r} = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{R}\frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{\infty}^{R} = \frac{1}{4\pi\varepsilon}\frac{q}{R}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\frac{q}{R} = \frac{\sigma R}{\varepsilon}}$$

(2) At internal point: The electric potential can be found by line integral of electric field over range ∞ to r ($<R$). i.e.

$$\phi = -\int_{\infty}^{r} \vec{E}.d\vec{r} = -\left[\int_{\infty}^{R}\vec{E}_{ext}.d\vec{r} + \int_{R}^{r}\vec{E}_{int}.d\vec{r}\right]$$

$$\therefore \quad \phi = -\left[\int_{\infty}^{R}\frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}.d\vec{r} + \int_{R}^{r}0.d\vec{r}\right] = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{R}\frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{\infty}^{R} = \frac{1}{4\pi\varepsilon}\frac{q}{R}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\frac{q}{R} = \frac{\sigma R}{\varepsilon}}$$

Electrostatic Potential due to uniformly charge sphere :

Suppose q charge is uniformly distributed over a sphere of radius of R. If ρ is the volume charge density of distribution then $q = \frac{4\pi}{3}R^3\rho$. The electric field at external and internal point due to this charge distribution is given by following expressions.

$$\vec{E}_{ext} = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r} \text{ and } \vec{E}_{int} = \frac{1}{4\pi\varepsilon}\frac{qr}{R^3}\hat{r}$$

The electrostatic potential due to this electrostatic system at external, surface and internal point can be obtained in following manner.

(2) At external point: The electric potential can be found by line integral of electric field over range ∞ to r ($>R$). i.e.

$$\because \quad \phi = -\int_{\infty}^{r} \vec{E}.d\vec{r}$$

$$\therefore \quad \phi = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}.d\vec{r} = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{r}\frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{\infty}^{r} = \frac{1}{4\pi\varepsilon}\frac{q}{r}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\frac{q}{r} = \frac{\rho}{3\varepsilon}\frac{R^3}{r}}$$

(3) At surface point: The electric potential can be found by line integral of electric field over range ∞ to R, i.e.

$$\phi = -\int_{\infty}^{R} \vec{E}.d\vec{r}$$

$$\therefore \quad \phi = -\int_{\infty}^{R} \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}.d\vec{r} = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{R}\frac{dr}{r^2} = -\frac{q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{\infty}^{R} = \frac{1}{4\pi\varepsilon}\frac{q}{R}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon}\frac{q}{R} = \frac{\rho R^2}{3\varepsilon}}$$

D K Pandey

(4) At internal point: The electric potential can be found by line integral of electric field over range ∞ to r ($<R$). i.e.

$$\phi = -\int_{\infty}^{r}\vec{E}.d\vec{r} = -\left[\int_{\infty}^{R}\vec{E}_{ext}.d\vec{r} + \int_{R}^{r}\vec{E}_{int}.d\vec{r}\right]$$

$$\therefore \qquad \phi = -\left[\int_{\infty}^{R}\frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}.d\vec{r} + \int_{R}^{r}\frac{1}{4\pi\varepsilon}\frac{qr}{R^3}\hat{r}.d\vec{r}\right] = -\frac{q}{4\pi\varepsilon}\left[\int_{\infty}^{R}\frac{dr}{r^2} + \frac{1}{R^3}\int_{R}^{r}r\,dr\right] = -\frac{q}{4\pi\varepsilon}\left[-\frac{1}{R} + \frac{(r^2-R^2)}{2R^3}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon}\left[\frac{1}{R} - \frac{(r^2-R^2)}{2R^3}\right] = \frac{q}{4\pi\varepsilon}\left[\frac{2R^2-r^2+R^2}{2R^3}\right] = \frac{q}{4\pi\varepsilon}\left[\frac{3R^2-r^2}{2R^3}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon}\left[\frac{3R^2-r^2}{2R^3}\right] = \frac{\rho}{4\pi\varepsilon}\frac{4\pi}{3}R^3\left[\frac{3R^2-r^2}{2R^3}\right] = \frac{\rho}{3\varepsilon}\left[\frac{3R^2-r^2}{2}\right]$$

$$\boxed{\phi = \frac{q}{4\pi\varepsilon}\left[\frac{3R^2-r^2}{2R^3}\right] = \frac{\rho}{3\varepsilon}\left[\frac{3R^2-r^2}{2}\right]}$$

Electrostatic potential energy: The work done to constitute an electrostatic system by bringing the charges from infinity is called as electrostatic potential energy.

$$u = \sum_i (W_{\infty\to r})_i = W_1 + W_2 + W_3 + \cdots$$

where $W_{\infty\to r} = -\int_{\infty}^{r}\vec{F}.d\vec{r}$ and $W_1, W_2, W_3, \cdots$ are the work done to bring first, second, third, ...charges respectively.

(1) Electrostatic system of single point charge: If a single point charge is brought from ∞ to some point then no work is done due to absence of no field and no force acts on it. i.e.

$$u = W = 0$$

(2) Electrostatic system of two point charge: Suppose two point charges q_1 and q_2 are brought from ∞ one by one to constitute a electrostatic system of two charges separated by distance r.

$$u = \sum_i (W_{\infty\to r})_i = W_1 + W_2 = 0 - \int_{\infty}^{r}\vec{F}.d\vec{r} = -\int_{\infty}^{r}\frac{1}{4\pi\varepsilon}\frac{q_1q_2}{r^2}\hat{r}.d\vec{r} = -\frac{q_1q_2}{4\pi\varepsilon}\int_{\infty}^{r}\frac{dr}{r^2} = -\frac{q_1q_2}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{\infty}^{r} = \frac{1}{4\pi\varepsilon}\frac{q_1q_2}{r}$$

$$\boxed{u = \frac{1}{4\pi\varepsilon}\frac{q_1q_2}{r}}$$

(3) Electrostatic system of three point charge in shape of triangle: Suppose three point charges q_1 , q_2 and q_3 are brought from ∞ one by one to constitute a electrostatic system of three charges in shape of triangle. The distance between q_1 and q_2 is r_1 while distance between q_2 and q_3 is r_2 and between q_3 and q_1 is r_3.

$$u = W_1 + W_2 + W_2 = 0 + \left(\frac{1}{4\pi\varepsilon}\frac{q_1q_2}{r_1}\right) + \left(\frac{1}{4\pi\varepsilon}\frac{q_1q_3}{r_3} + \frac{1}{4\pi\varepsilon}\frac{q_2q_3}{r_2}\right)$$

$$\boxed{u = \frac{1}{4\pi\varepsilon}\frac{q_1q_2}{r_1} + \frac{1}{4\pi\varepsilon}\frac{q_2q_3}{r_2} + \frac{1}{4\pi\varepsilon}\frac{q_3q_1}{r_3}}$$

(4) Electrostatic system of n point charges: Suppose 'n' numbers of point charges are brought from ∞ one by one and constitutes an electrostatic system. If The distance between charges q_i and q_j is r_{ij} then electrostatic potential energy can be written as,

$$\boxed{u = \frac{1}{2}\sum_{i,j=1}^{n}\frac{1}{4\pi\varepsilon}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{1}{4\pi\varepsilon}\frac{q_i}{r_{ij}}\right)q_j = \frac{1}{2}\sum_{i,j=1}^{n}\phi_iq_j}$$

D K Pandey

If the charges are uniformly distributed then summation sign is changed to integration sign. Hence equation of u becomes as,

$$\boxed{u = \frac{1}{2}\int \phi\, dq}$$

For line charge distribution, $u = \frac{1}{2}\int \phi\lambda\, dl$

For surface charge distribution, $u = \frac{1}{2}\int \phi\sigma\, ds$

For volume charge distribution, $u = \frac{1}{2}\int \phi\rho\, d\tau$

Electrostatic energy density:

The energy required to constitute an electrostatic system is called as electrostatic potential energy. This energy is stored in the electrostatic system. The electrostatic energy per unit volume is termed as electrostatic energy density.

$$U = \frac{du}{d\tau} \qquad (1)$$

The electrostatic energy of uniform volume charge distribution is given by following formula.

$$u = \frac{1}{2}\int \phi\rho\, d\tau \qquad (2)$$

According to Gauss law in differential form, we can write

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \Rightarrow \qquad \rho = \varepsilon_0 \vec{\nabla}.\vec{E} \qquad (3)$$

Substituting value of ρ *from equation (3) to equation (2), we have*

$$u = \frac{\varepsilon_0}{2}\int \phi(\vec{\nabla}.\vec{E})\, d\tau \qquad (4)$$

For vector identity, we can write that,

$$\vec{\nabla}.(\phi\vec{E}) = \phi(\vec{\nabla}.\vec{E}) + \vec{E}.(\vec{\nabla}\phi)$$

$$\vec{\nabla}.(\phi\vec{E}) = \phi(\vec{\nabla}.\vec{E}) - \vec{E}.\vec{E} \qquad (as\ \vec{E} = -\vec{\nabla}\phi)$$

$$\phi(\vec{\nabla}.\vec{E}) = \vec{\nabla}.(\phi\vec{E}) + E^2 \qquad (5)$$

From equations (4) and (5), we have

$$u = \frac{\varepsilon_0}{2}\int \left(\vec{\nabla}.(\phi\vec{E}) + E^2\right) d\tau = \frac{1}{2}\varepsilon_0 \int \vec{\nabla}.(\phi\vec{E})\, d\tau + \frac{1}{2}\varepsilon_0 \int E^2\, d\tau$$

$$u = \frac{1}{2}\varepsilon_0 \int \phi\vec{E}.\, d\vec{s} + \frac{1}{2}\varepsilon_0 \int E^2\, d\tau \qquad (6)$$

In the integrand of surface integral, first factor $\phi \propto \frac{1}{r}$, *second factor* $E \propto \frac{1}{r^2}$ *and third factor* $ds \propto r^2$, *thus integrand varies as* $1/r$. *So, if we integrate equation (6) over infinite volume, the first integral contributes nothing to energy and hence, energy of system becomes as,*

$$\boxed{u = \frac{1}{2}\varepsilon_0 \int E^2\, d\tau} \qquad (7)$$

Equation (7) can also be written as,

$$\int du = \int \left(\frac{1}{2}\varepsilon_0 E^2\right) d\tau \qquad \Rightarrow \qquad du = \frac{1}{2}\varepsilon_0 E^2\, d\tau \qquad \Rightarrow \qquad \frac{du}{d\tau} = \frac{1}{2}\varepsilon_0 E^2 \qquad (8)$$

From equation (1) and (8), we obtained that,

$$\boxed{U = \frac{1}{2}\varepsilon_0 E^2} \qquad (9)$$

D K Pandey

Electrostatic energy of uniformly charged spherical shell:

Suppose a charge q is uniformly distributed over a spherical shell of radius R. The surface charge density is $\sigma = q/4\pi R^2$. *The electrostatic energy of this charged spherical shell can be evaluated with following two methods.*

Method 1:

$$u = \frac{1}{2}\int_0^R \phi\sigma ds$$

$$u = \frac{1}{2}\int_0^R \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\right)\left(\frac{q}{4\pi R^2}\right)ds$$

$$u = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\right)\left(\frac{q}{4\pi R^2}\right)\int_0^R ds$$

$$u = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\right)\left(\frac{q}{4\pi R^2}\right)4\pi R^2$$

$$\boxed{u = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}}$$

Method 2:

$$u = \frac{1}{2}\varepsilon_0\int_0^\infty E^2\, d\tau = \frac{1}{2}\varepsilon_0\left[\int_0^R E^2\, d\tau + \int_R^\infty E^2\, d\tau\right]$$

$$u = \frac{1}{2}\varepsilon_0\left[\int_0^R 0\, d\tau + \int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2 \left(4\pi r^2 dr\right)\right]$$

$$u = \frac{1}{2}\varepsilon_0\left[0 + \frac{q^2}{4\pi\varepsilon_0\varepsilon_0}\int_R^\infty \frac{dr}{r^2}\right] = \frac{1}{2}\varepsilon_0\frac{q^2}{4\pi\varepsilon_0\varepsilon_0}\frac{1}{R}$$

$$\boxed{u = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}}$$

Electrostatic energy of uniformly charged sphere:

Suppose a charge q is uniformly distributed over a sphere of radius R. The volume charge density is $\rho = q/(4\pi/3)R^3$. *The electrostatic energy of this charged spherical shell can be evaluated with following two methods.*

Method 1:

$$u = \frac{1}{2}\int_0^R \phi\rho d\tau$$

$$u = \frac{1}{2}\int_0^R \left(\frac{q}{4\pi\varepsilon_0}\frac{(3R^2 - r^2)}{2R^3}\right)\frac{3q}{4\pi R^3}4\pi r^2 dr$$

$$u = \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\frac{3q}{4\pi R^3}\frac{4\pi}{2R^3}\int_0^R \left(3R^2 - r^2\right)r^2 dr$$

$$u = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{3}{2R^6}\int_0^R \left(3R^2 r^2 - r^4\right)dr$$

$$u = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{3}{2R^6}\left(3R^2\frac{R^3}{3} - \frac{R^5}{5}\right)$$

$$u = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{3}{2R^6}\frac{4R^5}{5}$$

$$\boxed{u = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{q^2}{R}}$$

Method 2:

$$u = \frac{1}{2}\varepsilon_0\int_0^\infty E^2\, d\tau = \frac{1}{2}\varepsilon_0\left[\int_0^R E^2\, d\tau + \int_R^\infty E^2\, d\tau\right]$$

$$u = \frac{1}{2}\varepsilon_0\left[\int_0^R \left(\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\right)^2\left(4\pi r^2 dr\right) + \int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2\left(4\pi r^2 dr\right)\right]$$

$$u = \frac{1}{2}\varepsilon_0\left[\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\right)^2(4\pi)\int_0^R r^4 dr + \left(\frac{q}{4\pi\varepsilon_0}\right)^2(4\pi)\int_R^\infty \frac{dr}{r^2}\right]$$

$$u = \frac{1}{2}\varepsilon_0\left[\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\right)^2(4\pi)\frac{R^5}{5} + \left(\frac{q}{4\pi\varepsilon_0}\right)^2(4\pi)\frac{1}{R}\right]$$

$$u = \frac{1}{2}\left[\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R}\right)\frac{1}{5} + \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R}\right)\right]$$

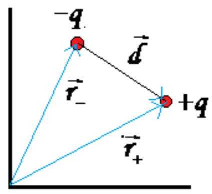
$$u = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R}\right)\frac{6}{5} \Rightarrow \boxed{u = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{q^2}{R}}$$

D K Pandey

Electric dipole: *A system of equal and opposite charges separated by small distance is called as 'electric dipole'. The product of the magnitude of any one charge and the distance between the charges is termed as 'electric dipole moment'. It is a vector quantity, whose direction is along the axis of dipole pointing from negative to positive charge. It is denoted by* $\vec{p}$.

The dipole moment of any electrostatic system of 'n' number of charges is equal to vector sum of product of each charges and their position vector. If q_1, q_2, q_3, …… *having position vectors* r_1, r_2, r_3, …….. *constitute an electrostatic system then its dipole moment can be written as,*

$$\vec{p} = q_1\vec{r}_1 + q_2\vec{r}_2 + q_3\vec{r}_3 + \cdots = \sum_{i=1}^{n} q_i\vec{r}_i$$



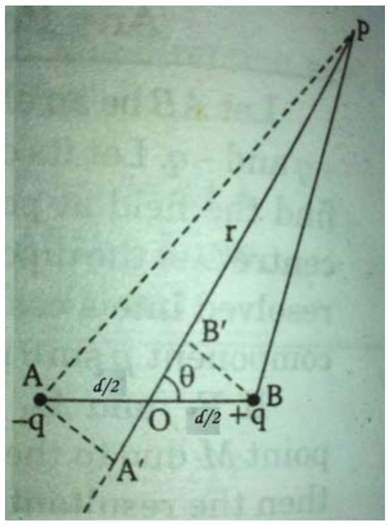
Suppose +q and –q charges have the position vectors r_+ *and* r_- . *These charges are separated by small distance d. The electric dipole moment of this system can be written as,*

$$\vec{p} = (+q)\vec{r}_+ + (-q)\vec{r}_- = q(\vec{r}_+ - \vec{r}_-) = q\,\vec{d}$$

Here $\vec{d}$ *is vector joining the negative to positive charge.*

Electrostatic potential and electric field due electric dipole:

Method 1: *Suppose a charges –q and +q are placed at points A and B which are separated by a small distance 'd'. P is point which is situated at distance 'r' from the centre of dipole and at inclination 'θ' with dipole axis. Let* ϕ_1 *and* ϕ_2 *are the potentials at point P due to point charges –q and +q charges respectively. Then the total potential at point P can be written as scalar sum of them. i.e.*



$$\phi = \phi_1 + \phi_2$$

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{PA} + \frac{1}{4\pi\varepsilon_0}\frac{(+q)}{PB}$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{PA} + \frac{1}{PB}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{PB} - \frac{1}{PA}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{PB'} - \frac{1}{PA'}\right] \qquad (1)$$

From figure,

$$PA = PA' = r + \frac{d}{2}\cos\theta$$

$$PB = PB' = r - \frac{d}{2}\cos\theta$$

Thus from equation (1), we can write,

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left(r - \frac{d}{2}\cos\theta\right)} - \frac{1}{\left(r + \frac{d}{2}\cos\theta\right)}\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{r + \frac{d}{2}\cos\theta - r + \frac{d}{2}\cos\theta}{\left(r^2 - \frac{d^2}{4}\cos^2\theta\right)}\right]$$

D K Pandey

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{\left(r^2 - \frac{d^2}{4}\cos^2\theta\right)}\right] \qquad (2)$$

As the dipole is short, so $r^2 >> d^2$ hence second term in the denominator can be neglected in comparison with r^2. Thus equation (2) becomes as,

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{qd\cos\theta}{r^2}\right]$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}} \qquad (3)$$

Equation (2) is expression of electrostatic potential due to electric dipole.

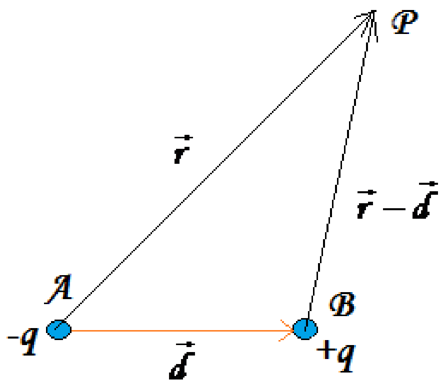
at axial point, $\theta = 0$, *hence,* $\phi = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}$

at transverse point, $\theta = 90$, *hence,* $\phi = 0$

and if $\theta = 180$, *hence,* $\phi = -\frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}$

Thus potential is maximum along the line joining the charges. At one side it is positive and on other side, it is negative

Method 2: *Suppose a charges –q and +q are placed at points A and B which are separated by a small distance 'd'. The position vector of charge +q w.r.t. charge –q is $\vec{d}$. P is point whose position vector w.r.t. charge –q is $\vec{r}$. Thus the position vector of P w.r.t. charge +q will be $\vec{r} - \vec{d}$. Let ϕ_1 and ϕ_2 are the potentials at point P due to point charges –q and +q charges respectively. Then*



$$\phi = \phi_1 + \phi_2$$

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|\vec{r}|} + \frac{1}{4\pi\varepsilon_0}\frac{(+q)}{|\vec{r}-\vec{d}|}$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{d}|} - \frac{1}{|\vec{r}|}\right] \qquad (1)$$

Now, $$\frac{1}{|\vec{r}-\vec{d}|} = \frac{1}{\left\{(\vec{r}-\vec{d}).(\vec{r}-\vec{d})\right\}^{1/2}}$$

$$\frac{1}{|\vec{r}-\vec{d}|} = \left\{(\vec{r}-\vec{d}).(\vec{r}-\vec{d})\right\}^{-1/2}$$

$$\frac{1}{|\vec{r}-\vec{d}|} = \left\{r^2 + d^2 - 2\vec{r}.\vec{d}\right\}^{-1/2} = \frac{1}{r}\left\{1 + \frac{d^2}{r^2} - \frac{2\vec{r}.\vec{d}}{r^2}\right\}^{-1/2}$$

As the dipole is short, $r^2 >> d^2$. Hence, d^2/r^2 can be neglected.

$$\frac{1}{|\vec{r}-\vec{d}|} = \frac{1}{r}\left\{1 - \frac{2\vec{r}.\vec{d}}{r^2}\right\}^{-1/2} = \frac{1}{r}\left\{1 + \frac{\vec{r}.\vec{d}}{r^2}\right\} \qquad (2)$$

D K Pandey

From equations (1) and (2), we can write,

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left\{1+\frac{\vec{r}.\vec{d}}{r^2}\right\}-\frac{1}{|\vec{r}|}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}+\frac{\vec{r}.\vec{d}}{r^3}-\frac{1}{r}\right]$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}.\vec{d}}{r^3}=\frac{1}{4\pi\varepsilon_0}\frac{q\vec{d}.\vec{r}}{r^3}$$

$$\boxed{\phi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{r}}{r^3}=\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^3}} \qquad (3)$$

here θ is angle between vectors $\vec{p}$ and $\vec{r}$.

Electric field due electric dipole:

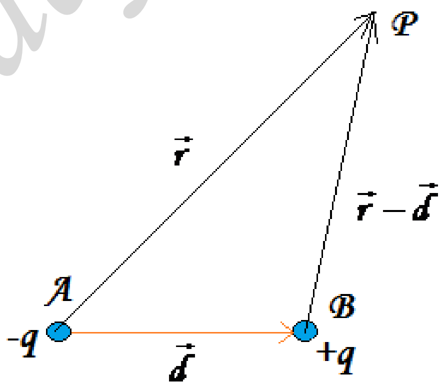
Method 1: *Suppose a charges –q and +q are placed at points A and B which are separated by a small distance 'd'. The position vector of charge +q w.r.t. charge –q is $\vec{d}$. P is point whose position vector w.r.t. charge –q is $\vec{r}$. Thus the position vector of P w.r.t. charge +q will be $\vec{r}-\vec{d}$. Let $\vec{E}_1$ and $\vec{E}_2$ are the potentials at point P due to point charges –q and +q charges respectively. Then*

$$\vec{E}=\vec{E}_1+\vec{E}_2$$

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{q(-\vec{r})}{|\vec{r}|^3}+\frac{1}{4\pi\varepsilon_0}\frac{q(\vec{r}-\vec{d})}{|\vec{r}-\vec{d}|^3}$$

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[-\frac{\vec{r}}{|\vec{r}|^3}+\frac{(\vec{r}-\vec{d})}{|\vec{r}-\vec{d}|^3}\right]$$

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{(\vec{r}-\vec{d})}{|\vec{r}-\vec{d}|^3}-\frac{\vec{r}}{|\vec{r}|^3}\right] \qquad (1)$$



Now, $$\frac{1}{|\vec{r}-\vec{d}|^3}=\frac{1}{\{(\vec{r}-\vec{d}).(\vec{r}-\vec{d})\}^{3/2}}$$

$$\frac{1}{|\vec{r}-\vec{d}|^3}=\{(\vec{r}-\vec{d}).(\vec{r}-\vec{d})\}^{-3/2}=\{r^2+d^2-2\vec{r}.\vec{d}\}^{-3/2}=\frac{1}{r^3}\left\{1+\frac{d^2}{r^2}-\frac{2\vec{r}.\vec{d}}{r^2}\right\}^{-3/2}$$

As the dipole is short, $r^2 >> d^2$. Hence, d^2/r^2 can be neglected.

$$\frac{1}{|\vec{r}-\vec{d}|^3}=\frac{1}{r^3}\left\{1-\frac{2\vec{r}.\vec{d}}{r^2}\right\}^{-3/2}=\frac{1}{r^3}\left\{1+\frac{3\vec{r}.\vec{d}}{r^2}\right\}=\frac{1}{r^3}\left\{1+\frac{3\vec{d}.\vec{r}}{r^2}\right\} \qquad (2)$$

From equations (1) and (2), we can write,

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r^3}\left\{1+\frac{3\vec{d}.\vec{r}}{r^2}\right\}(\vec{r}-\vec{d})-\frac{\vec{r}}{r^3}\right]=\frac{q}{4\pi\varepsilon_0}\left[\left\{\frac{1}{r^3}+\frac{3\vec{d}.\vec{r}}{r^5}\right\}(\vec{r}-\vec{d})-\frac{\vec{r}}{r^3}\right]$$

D K Pandey

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{r}}{r^3}+\frac{3(\vec{d}.\vec{r})\vec{r}}{r^5}-\frac{\vec{d}}{r^3}-\frac{3\vec{d}.\vec{r}}{r^5}\vec{d}-\frac{\vec{r}}{r^3}\right]$$

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{3(\vec{d}.\vec{r})\vec{r}}{r^5}-\frac{\vec{d}}{r^3}-\frac{3\vec{d}.\vec{r}}{r^5}\vec{d}\right] \qquad (3)$$

In the equation (3), third term is too much small in comparison with other two term, thus can be neglected. Hence equation (3) becomes as,

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{3(\vec{d}.\vec{r})\vec{r}}{r^5}-\frac{\vec{d}}{r^3}\right]=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(q\vec{d}.\vec{r})\vec{r}}{r^5}-\frac{q\vec{d}}{r^3}\right]=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}.\vec{r})\vec{r}}{r^5}-\frac{\vec{p}}{r^3}\right]$$

$$\boxed{\vec{E}=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{r}.\vec{p})\,\vec{r}}{r^5}-\frac{\vec{p}}{r^3}\right]} \qquad (4)$$

Equation (4) is expression of electric field due to an electric dipole.

Method 2: *Suppose a charges –q and +q are placed at points A and B which are separated by a small distance 'd'. The position vector of charge +q w.r.t. charge –q is $\vec{d}$. P is point whose position vector w.r.t. charge –q is $\vec{r}$. Thus the position vector of P w.r.t. charge +q will be $\vec{r}-\vec{d}$. If ϕ is the potential at point P due to the electric dipole then it can be given by following expression.*

$$\phi=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{r}}{r^3} \qquad (1)$$

The electric field and potential can be co-related as,

$$\vec{E}=-\vec{\nabla}\phi \qquad (2)$$

From equations (1) and (2), we can write.

$$\vec{E}=-\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{\vec{p}.\vec{r}}{r^3}\right)=-\frac{1}{4\pi\varepsilon_0}\left[(\vec{p}.\vec{r})\vec{\nabla}\left(\frac{1}{r^3}\right)+\frac{1}{r^3}\vec{\nabla}(\vec{p}.\vec{r})\right] \qquad (3)$$

Since $\vec{\nabla}(\vec{p}.\vec{r})=\vec{p}$ *and* $\vec{\nabla}\left(\frac{1}{r^3}\right)=-3\frac{\vec{r}}{r^5}$, *thus putting values in equation (3) we have,*

$$\vec{E}=-\frac{1}{4\pi\varepsilon_0}\left[(\vec{p}.\vec{r})\left(-3\frac{\vec{r}}{r^3}\right)+\frac{\vec{p}}{r^3}\right]$$

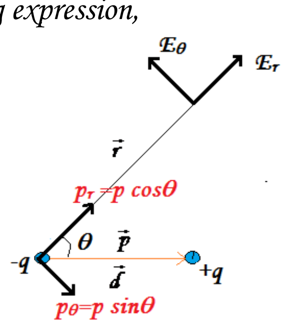
$$\boxed{\vec{E}=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{r}.\vec{p})\,\vec{r}}{r^5}-\frac{\vec{p}}{r^3}\right]} \qquad (4)$$

Equation (4) is expression of electric field due to an electric dipole.

Polar component of electric field and total elelctric field due to electric dipole:

Method 1: *The electric field due electric dipole is given by following expression,*

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{r}.\vec{p})\,\vec{r}}{r^5}-\frac{\vec{p}}{r^3}\right] \qquad (a)$$



The radial component of equation (a) can be written as,

$$E_r=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(r\,p\cos\theta)\,r}{r^5}-\frac{p\cos\theta}{r^3}\right]=\frac{1}{4\pi\varepsilon_0}\left[\frac{3\,p\cos\theta}{r^3}-\frac{p\cos\theta}{r^3}\right]$$

$$E_r=\frac{1}{4\pi\varepsilon_0}\frac{2\,p\cos\theta}{r^3} \qquad (b)$$

D K Pandey

The transverse component of equation (a) can be written as,

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3} \qquad (c)$$

Thus the total electric field due to electric dipole at an arbitrary point P(r,θ) can be obtained as,

$$E = \sqrt{E_r^2 + E_\theta^2} = \sqrt{\left(\frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}\right)^2 + \left(\frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}\right)^2} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta + 1}} \qquad (d)$$

The equation (d) is expression of electric field due to an electric dipole.

$$If\ \theta = 0^o\ then\ E = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}$$

$$If\ \theta = 90^o\ then\ E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}$$

Method 2:

The electric potential due to an electric dipole at an arbitrary point P(r,θ) is given by following expression,

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \qquad (a)$$

Since the electric field and potential can be co-related as,

$$\vec{E} = -\vec{\nabla}\phi$$

Thus the radial and transverse component of electric field can be written as,

$$E_r = -\frac{\partial\phi}{\partial r}\ and\ E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

Using equation (a), we find that,

$$E_r = -\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right) = -\frac{p\cos\theta}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) = -\frac{p\cos\theta}{4\pi\varepsilon_0}\left(-\frac{2}{r^3}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3} \qquad (b)$$

And $$E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right) = -\frac{p}{4\pi\varepsilon_0 r^3}\frac{\partial}{\partial r}(\cos\theta) = -\frac{p}{4\pi\varepsilon_0 r^3}(-\sin\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3} \qquad (c)$$

Thus the total electric field due to electric dipole at an arbitrary point P(r,θ) can be obtained as,

$$E = \sqrt{E_r^2 + E_\theta^2} = \sqrt{\left(\frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}\right)^2 + \left(\frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}\right)^2} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta + 1}} \qquad (d)$$

The equation (d) is expression of electric field due to an electric dipole.

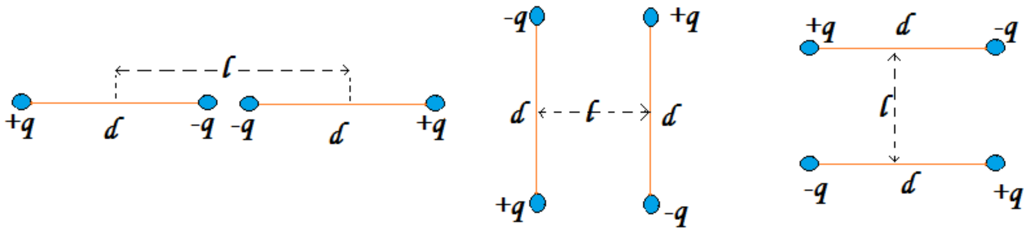
$$If\ \theta = 0^o\ then\ E = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}$$

$$If\ \theta = 90^o\ then\ E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}$$

D K Pandey

Electric Quadrupole and Quadrupole moment:

An electric quadrupole is an arrangement which consists of two equal and opposite dipoles separated by small distance. i.e. it is an arrangement of two parallel dipoles of opposite polarity. Both the dipoles do not coincide in space such that their electric effects do not cancel out at distant point.
Following are the some quadrupole arrangements.



(i) (ii) (iii)

The quadrupole moment of a charge distribution is defined as,

$$Q_d = \int \frac{\rho(r')r'^2}{2}\left(3\cos^2\theta - 1\right)dv'$$

If point charges are distributed in space then quadrupole moment of distribution can be written as,

$$Q_d = \sum_i \frac{q_i r_i'^2}{2}\left(3\cos^2\theta - 1\right)$$

Suppose a linear quadrupole is formed by charges +q, -2q and +q (Ist arrangement) whose position co-ordinates aree (d,0), (0,0) and (-d, 0) respectively. The quadrupole moment of linear quadrupole at external point P (which have θ inclination from axis of quadrupole) can be obtained as,

$$Q_d = \frac{q(d)^2}{2}\left(3\cos^2\theta - 1\right) + \frac{(-2q)(0)^2}{2}\left(3\cos^2\theta - 1\right) + \frac{q(-d)^2}{2}\left(3\cos^2(180-\theta) - 1\right)$$

$$Q_d = \frac{q\,d^2}{2}\left(3\cos^2\theta - 1\right) + 0 + \frac{q\,d^2}{2}\left(3\cos^2\theta - 1\right)$$

$$\boxed{Q_d = q\,d^2\left(3\cos^2\theta - 1\right)}$$

If external point P have 0 degree inclination from axis of quadrupole then,

$$\boxed{Q_d = Q_0 = q\,d^2\left(3\cos^2 0 - 1\right) = q\,d^2(3-1) = 2q\,d^2}$$

Therefore, the quadrupole moment of a linear quadrupole at its axial point is twice times product of single charge and square of distance between two charges in any of dipole.

Electrostatic potential and electric field due electric quadrupole:

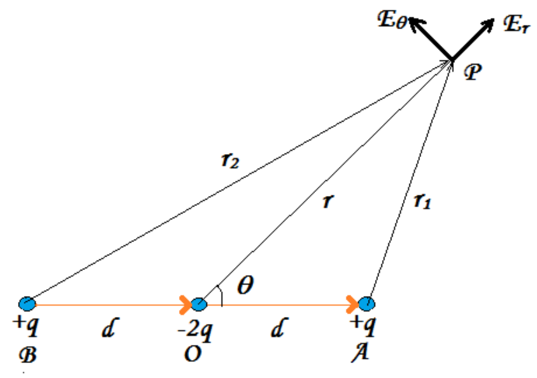
Suppose a charges +q, -2q and +q form a linear quadrupole which are placed at points A, O and B along x-axis. P is point which is situated at distance 'r' from the centre of quadrupole while Charges placed at A and B are distance r_1 and r_2 respectively. Let ϕ_1 , ϕ_2 and ϕ_3 are the potentials at point P due to point charges +q, -2q and +q respectively. Then the total potential at point P can be written as scalar sum of them. i.e.

$$\phi = \phi_1 + \phi_2 + \phi_3$$

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{(+q)}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{(-2q)}{r} + \frac{1}{4\pi\varepsilon_0}\frac{(+q)}{r_1}$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} + +\frac{1}{r_2} - \frac{2}{r}\right] \qquad (1)$$

From figure we can write,



$$\vec{r}_1 = \vec{r} - \vec{d}$$
$$\vec{r}_1.\vec{r}_1 = (\vec{r} - \vec{d}).(\vec{r} - \vec{d})$$
$$r_1^2 = r^2 + d^2 - 2r\,d\,\cos\theta$$
$$r_1 = (r^2 + d^2 - 2r\,d\,\cos\theta)^{1/2}$$
$$\frac{1}{r_1} = (r^2 + d^2 - 2r\,d\,\cos\theta)^{-1/2}$$
$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right)^{-1/2}$$
$$\frac{1}{r_1} = \frac{1}{r}\left[1 + \left(\frac{d^2}{r^2} - \frac{2d}{r}\cos\theta\right)\right]^{-1/2} \qquad (2)$$

Expanding the right side of equation (2) and neglecting higher order terms under condition $r >> d$ we have,

$$\frac{1}{r_1} = \frac{1}{r}\left[1 + \frac{d^2}{2r^2}(3\cos^2\theta - 1) + \frac{d}{r}\cos\theta\right] \qquad (3)$$

Similarly, from figure we can write,

$$\vec{r}_2 = \vec{r} + \vec{d}$$
$$\vec{r}_2.\vec{r}_2 = (\vec{r} + \vec{d}).(\vec{r} + \vec{d})$$
$$r_2^2 = r^2 + d^2 + 2r\,d\,\cos\theta$$
$$r_2 = (r^2 + d^2 + 2r\,d\,\cos\theta)^{1/2}$$
$$\frac{1}{r_2} = (r^2 + d^2 + 2r\,d\,\cos\theta)^{-1/2}$$
$$\frac{1}{r_2} = \frac{1}{r}\left(1 + \frac{d^2}{r^2} + \frac{2d}{r}\cos\theta\right)^{-1/2}$$
$$\frac{1}{r_2} = \frac{1}{r}\left[1 + \left(\frac{d^2}{r^2} + \frac{2d}{r}\cos\theta\right)\right]^{-1/2} \qquad (4)$$

Expanding the right side of equation (4) and neglecting higher order terms under condition $r >> d$ we have,

$$\frac{1}{r_2} = \frac{1}{r}\left[1 + \frac{d^2}{2r^2}(3\cos^2\theta - 1) - \frac{d}{r}\cos\theta\right] \qquad (5)$$

From equations (1), (3) and (5), we have,

$$\phi = \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 + \frac{d^2}{2r^2}(3\cos^2\theta - 1) + \frac{d}{r}\cos\theta\right) + \left(1 + \frac{d^2}{2r^2}(3\cos^2\theta - 1) - \frac{d}{r}\cos\theta\right) - 2\right]$$
$$\phi = \frac{q}{4\pi\varepsilon_0 r}\frac{d^2}{r^2}(3\cos^2\theta - 1)$$

D K Pandey

$$\boxed{\phi = \frac{1}{4\pi\varepsilon_0}\frac{qd^2}{r^3}\left(3\cos^2\theta - 1\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{2r^3}\left(3\cos^2\theta - 1\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q_d}{r^3}} \qquad (6)$$

Equation (6) is expression of potential due to linear quadrupole.

Since, $E_r = -\frac{\partial\phi}{\partial r}$ and $E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

Thus, $$E_r = -\frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_0}\frac{Q_0}{2r^3}\left(3\cos^2\theta - 1\right)\right]$$

$$E_r = -\frac{Q_0}{4\pi\varepsilon_0}\frac{\left(3\cos^2\theta - 1\right)}{2}\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right) = -\frac{Q_0}{4\pi\varepsilon_0}\frac{\left(3\cos^2\theta - 1\right)}{2}\left(\frac{-3}{r^4}\right)$$

$$\Rightarrow \quad \boxed{E_r = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}\left(3\cos^2\theta - 1\right)}$$

And $$E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{4\pi\varepsilon_0}\frac{Q_0}{2r^3}\left(3\cos^2\theta - 1\right)\right]$$

$$E_\theta = -\frac{1}{4\pi\varepsilon_0}\frac{Q_0}{2r^4}\frac{\partial}{\partial\theta}\left(3\cos^2\theta - 1\right)$$

$$\boxed{E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}2\cos\theta\sin\theta}$$

We know that, $E = \sqrt{E_r^2 + E_\theta^2}$

Hence, $$E = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}\sqrt{\left(3\cos^2\theta - 1\right)^2 + \left(2\cos\theta\sin\theta\right)^2}$$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}\sqrt{5\cos^4\theta - 2\cos^2\theta + 1}} \qquad (7)$$

Equation (7) is expression of electric field due to linear quadrupole.

(i) If point P lies on the axis of quadrupole: $\theta = 0$ or 180

$$E_{max} = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}\sqrt{5-2+1} \quad \Rightarrow \quad \boxed{E_{max} = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{r^4}}$$

(ii) If point P lies on the line perpendicular to the axis of quadrupole: $\theta = 90$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}\sqrt{0-0+1} \quad \Rightarrow \quad \boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{3Q_0}{2r^4}}$$

Note: **Monople, dipole and quadrupole moments:**
The monopole, dipole and quadrupole moment of a uniform charge distribution can be determined by following expressions.

Monopole moment $= \int \rho(r')\,dv'$

dipole moment $= \vec{p} = \int \rho(r')\vec{r}'\,dv'$

Quadrupole moment $= Q_d = \int \frac{\rho(r')r'^2}{2}\left(3\cos^2\theta - 1\right)dv'$

For non-uniform charge distribution, monopole, dipole and quadrupole moment can be determined by,

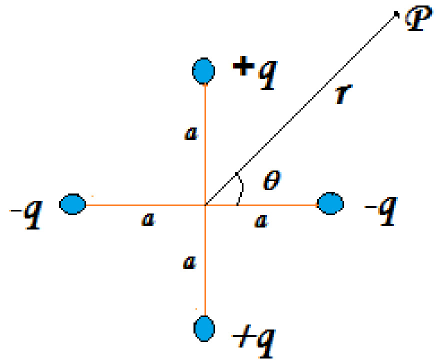
Monopole moment $= \sum_i q_i$

dipole moment $= \vec{p} = \sum_i q_i\vec{r}_i$

Quadrupole moment $= Q_d = \sum_i \frac{q_i r_i'^2}{2}\left(3\cos^2\theta - 1\right)$

D K Pandey

Example: *Find monopole, dipole and quadrupole moment of following charge distribution. Also find field at point P located at (r, θ) poition.*



Soluton: *For non-uniform charge distribution, monopole, dipole and quadrupole moment can be determined by,*

Monopole moment$=\sum_i q_i = -q + q - q + q = 0$

dipole moment$=\vec{p} = \sum_i q_i \vec{r}_i$

$$\vec{p} = (-q)\, a\,\hat{i} + (+q)\, a\hat{j} + (-q)\ (-a\hat{i}) + (+q)\ (-a\hat{j})$$

$$\vec{p} = -q\, a\,\hat{i} + q\, a\,\hat{j} + q\, a\,\hat{i} - q\ a\,\hat{j} = 0$$

Quadrupole moment$=Q_d = \sum_i \frac{q_i r_i'^2}{2}\left(3\cos^2\theta - 1\right)$

The contribution to quadrupole moment of charge –q at (a,0)

$$Q_1 = \frac{(-q)\, a^2}{2}\left(3\cos^2\theta - 1\right) = -\frac{q\, a^2}{2}\left(3\cos^2\theta - 1\right)$$

The contribution to quadrupole moment of charge +q at (0,a)

$$Q_2 = \frac{q\,(-a)^2}{2}\left(3\cos^2(90-\theta) - 1\right) = +\frac{q\, a^2}{2}\left(3\sin^2\theta - 1\right)$$

The contribution to quadrupole moment of charge –q at (-a,0)

$$Q_3 = \frac{(-q)\,(-a)^2}{2}\left(3\cos^2(180-\theta) - 1\right) = -\frac{q\, a^2}{2}\left(3\cos^2\theta - 1\right)$$

The contribution to quadrupole moment of charge +q at (0, -a)

$$Q_4 = \frac{q\,(-a)^2}{2}\left(3\cos^2(90+\theta) - 1\right) = +\frac{q\, a^2}{2}\left(3\sin^2\theta - 1\right)$$

Thus net quadrupole moment at pont P can be written as,

$$Q_d = Q_1 + Q_2 + Q_3 + Q_4$$

$$Q_d = -\frac{q\, a^2}{2}\left(3\cos^2\theta - 1\right) + \frac{q\, a^2}{2}\left(3\sin^2\theta - 1\right) - \frac{q\, a^2}{2}\left(3\cos^2\theta - 1\right) + \frac{q\, a^2}{2}\left(3\sin^2\theta - 1\right)$$

$$Q_d = -q\, a^2\left(3\cos^2\theta - 1\right) + q\, a^2\left(3\sin^2\theta - 1\right)$$

$$Q_d = q\, a^2\left\{-3\cos^2\theta + 1 + 3\sin^2\theta - 1\right\}$$

$$Q_d = 3q\, a^2\left\{\sin^2\theta - \cos^2\theta\right\}$$

$$Q_d = 3q\, a^2\left\{\sin^2\theta - 1 + \sin^2\theta\right\}$$

$$Q_d = 3q\, a^2\left\{2\sin^2\theta - 1\right\}$$

Since monopole and dipole moments are zero while quadrupole moment is non-zero thus potential at point P will be due to quadrupole,

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{3q\, a^2}{r^3}\left\{2\sin^2\theta - 1\right\}$$

Now radial $\vec{E}_r$ *and transverse* $\vec{E}_\theta$ *component of electric fields are defined by*

$$\vec{E}_r = -\frac{\partial\phi}{\partial r}\hat{r} \ \text{ and } \vec{E}_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

D K Pandey

And $\vec{E}_\theta = -\frac{3q\,a^2}{4\pi\varepsilon_0 r^4}\frac{\partial}{\partial r}\left(2\sin^2\theta - 1\right)\hat{\theta} = -\frac{3q\,a^2}{4\pi\varepsilon_0 r^4}4\sin\theta\cos\theta\;\hat{\theta} = -\frac{1}{4\pi\varepsilon_0}\frac{6q\,a^2}{r^4}\sin 2\theta\;\hat{\theta}$

Therefore, $\vec{E} = \vec{E}_r + \vec{E}_\theta \quad\Rightarrow\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{9q\,a^2}{r^4}\right)\left(2\sin^2\theta - 1\right)\hat{r} + -\frac{1}{4\pi\varepsilon_0}\frac{6q\,a^2}{r^4}\sin 2\theta\;\hat{\theta}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{3q\,a^2}{r^4}\right)\left[3\left(2\sin^2\theta - 1\right)\hat{r} - 2\sin 2\theta\;\hat{\theta}\right]$$

Example: *Find the component of electric field at a point if potential is given by* $\phi = 3x\,y^2 - x^3 + x\,z$.

Solution: *The relation in electric field and potential is* $\vec{E} = -\vec{\nabla}\phi$

Thus, $E_x = -\frac{\partial\phi}{\partial x} = -\frac{\partial}{\partial x}\left(3x\,y^2 - x^3 + x\,z\right) = -3y^2 + 3x^2 - z$

$E_y = -\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}\left(3x\,y^2 - x^3 + x\,z\right) = -6\,x\,y$ *and* $E_z = -\frac{\partial\phi}{\partial z} = -\frac{\partial}{\partial z}\left(3x\,y^2 - x^3 + x\,z\right) = -x$

Example: *Show that* $\phi = (A/r) + B$ *satisfies the Laplace equation. Here A and B are constants and r is magnitude of position vector* $\vec{r}$.

Solution: *Hint,* $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ *and* $r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{A}{r^3} + \frac{3A\,x^2}{r^5};\quad \frac{\partial^2\phi}{\partial y^2} = -\frac{A}{r^3} + \frac{3A\,y^2}{r^5}\quad\text{and}\quad \frac{\partial^2\phi}{\partial z^2} = -\frac{A}{r^3} + \frac{3A\,z^2}{r^5}$$

So, $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{3A}{r^3} + \frac{3A\left(x^2 + y^2 + z^2\right)}{r^5} = -\frac{3A}{r^3} + \frac{3A\,r^2}{r^5} = 0$, *hence proved.*

Note 1: *Energy stored in capacitor or condenser can be obtained in following ways.*

Work done to charge the capacitor by amount dq=$dW = V\,dq = \frac{q}{C}dq$

Thus, $W = \int_0^q \frac{q}{C}dq = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}C\,V^2$ *as,* $q = C\,V$

Energy stored per unit volume= $U = \frac{W}{volume} = \frac{1}{2}\frac{C\,V^2}{volume}$

For parallel plate capacitor, $C = \frac{\varepsilon A}{d}$, $V = E\,d$ *and volume=A d*

Hence, $U = \frac{1}{2}\frac{\varepsilon A}{d}\frac{(Ed)^2}{Ad} \quad\Rightarrow\quad \boxed{U = \frac{1}{2}\varepsilon E^2}$

Note 2: *Torque acting on electric dipole in an electric field: When an electric dipole is placed in an electric field, the charges +q and –q experience forces along and opposite to applied field. As a result torque acts on dipole which tries to rotate to make the energy minimum. The expression of torque and its energy in external electric field are given by following expressions.*

$\boxed{\vec{\tau} = \vec{p}\times\vec{E}}$ *and* $\boxed{u = -\vec{p}\cdot\vec{E}}$

Note 3: *When an electric field dipole is placed in field other electric dipole there is interaction between them. If electric dipole having dipole moment* $\vec{p}_1$ *placed in electric field* $(\vec{E}_2)$ *of dipole having moment* $\vec{p}_2$ *then interaction energy between them can be obtained as.*

$$u = -\vec{p}_1\cdot\vec{E}_2 = -\vec{p}_1\cdot\left[\frac{1}{4\pi\varepsilon_0}\left\{\frac{3\left(\vec{p}_2\cdot\vec{r}\right)\vec{r}}{r^5} - \frac{\vec{p}_2}{r^3}\right\}\right] \quad\Rightarrow\quad \boxed{u = \frac{1}{4\pi\varepsilon_0}\left\{\frac{\vec{p}_1\cdot\vec{p}_2}{r^3} - \frac{3\left(\vec{p}_1\cdot\vec{r}\right)\left(\vec{p}_2\cdot\vec{r}\right)}{r^5}\right\}}$$

D K Pandey