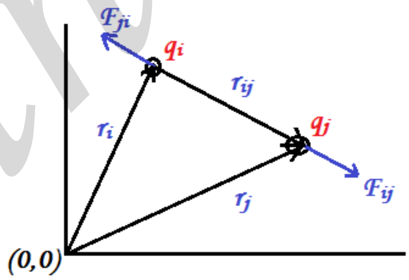
CSJM University   Class: B.Sc.-II   Sub:Physics   Paper-II
Title: Electromagnetics   Unit-1: Electrostatics   Lecture: 1 to 4

Electrostatics: *It deals the study of behavior of static or stationary Charges.*

Electric Charge: *It is property by which material get electrified. It is denoted by 'Q or q' and its unit is coulomb. Charges are of two types -(i) positive (ii) negative.*
A point charge is the dimensionless and mass less charge that can experience or exert electrostatic force. The point charge may be positive or negative. Unit positive charge which can only experience the electrostatic force is termed as 'test charge' and it is denoted by q_0.

Quantization of Charge: *The charge on material /matter/particle is integral multiple of charge on electron. i.e. q=±ne, where 'e' is charge on electron whose value is* 1.6×10^{-19} *coulomb. The charges on quarks are the exception to quantization of charge. Quarks may have charges* $\pm\frac{1}{3}e$ *or* $\pm\frac{2}{3}e$.

Coulombs Law: *The force acting between charges is called as electrostatic force. Coulombs law provides the magnitude and direction of electrostatic force acting between two charges. According to this law- "The electrostatic force between two charges is directly proportional to product of both charges and inversely proportional to square of distance between them".*
Let q_i *and* q_j *are the two point charges and are placed at a distance* r_{ij}.



$$F \propto \frac{q_i q_j}{r_{ij}^2}$$

or
$$\boxed{F = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ij}^2}} \qquad (1)$$

Here ε *is the permittivity of medium where the charges are placed. If* ε_0 *and* ε_r *are the permittivity of free space and relative permittivity or dielectric constant respectively then* ε *can be given by following expression.*

$\varepsilon = \varepsilon_0\varepsilon_r$; *here* $\varepsilon_0 = 8.854\times10^{-12}\,Coulomb^2/Nm^2$

In vector form the force is given by,

$$\vec{F} = F\hat{n} = F\,\hat{r}_{ij} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ij}^3}\vec{r}_{ij} \qquad (2)$$

This force is experienced by both the charges. Let $\vec{F}_{ij}$ *is force exerted by charge* q_i *on charge* q_j *and* $\vec{F}_{ji}$ *is force exerted by charge* q_j *on charge* q_i.

From equation (2) we can write,

$$\vec{F}_{ij} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ij}^3}\vec{r}_{ij} \qquad (3)$$

And
$$\vec{F}_{ji} = \frac{1}{4\pi\varepsilon}\frac{q_j q_i}{r_{ji}^2}\hat{r}_{ji} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ji}^3}\vec{r}_{ji} \qquad (4)$$

Since, $\vec{r}_{ij} = -\vec{r}_{ji}$ and $|\vec{r}_{ij}| = |-\vec{r}_{ji}|$ *thus from equations (3) and (4) we can have*

$$\boxed{\vec{F}_{ij} = -\vec{F}_{ji}} \qquad (5)$$

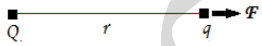
Equations (5) implies that the forces experienced by both charges are opposite in nature while their magnitudes are same.

Note 1: If $\vec{r}_i$ and $\vec{r}_j$ are the position vectors of q_i and q_j charges then electrostatic force is written as:

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{r_{ij}^3}\vec{r}_{ij} = \frac{1}{4\pi\varepsilon}\frac{q_i q_j}{\left|\vec{r}_j - \vec{r}_i\right|^3}\left(\vec{r}_j - \vec{r}_i\right)$$

Note 2: If Q and q charges are placed at a distant r then electrostatic force is written as:

$$F = \frac{1}{4\pi\varepsilon}\frac{Qq}{r^2} \quad and \quad \vec{F} = \frac{1}{4\pi\varepsilon}\frac{Qq}{r^3}\vec{r}$$



Superposition Principle: *The resultant electrostatic force on a charge is vector sum all forces experienced by it.*

i.e. $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots\cdots$

If q charge is surrounded by charges $q_1, q_2, q_3, \ldots$, which are at $r_1, r_2, r_3, \ldots$ apart from q respectively then,

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{q_1 q}{r_1^3}\vec{r}_1 + \frac{1}{4\pi\varepsilon}\frac{q_2 q}{r_2^3}\vec{r}_2 + \frac{1}{4\pi\varepsilon}\frac{q_3 q}{r_3^3}\vec{r}_3 + \cdots\cdots = \frac{q}{4\pi\varepsilon}\sum_i \frac{q_i}{r_i^3}\vec{r}_i$$

Electric Field: *The space or region around the charge or group of charges where the effect of these charges is felt or another charge experiences the force or is called as Electric field. It is denoted by $\vec{E}$. It is ratio of force experienced another charge or test charge and its amount of charge.*

i.e.
$$\boxed{\vec{E} = \frac{\vec{F}}{q_0}} \qquad (1)$$

(A) Electric Field due to point charge 'q':

Let q_0 charge is positioned in the electric field of charge q at distance 'r'. The force can be written as,

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{q\,q_0}{r^3}\vec{r} \qquad (2)$$

From equations (1) and (2), we find that,

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon}\frac{q}{r^3}\vec{r}} \quad and \quad \boxed{E = \left|\vec{E}\right| = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}} \qquad (3)$$

Equation (3) is expression for electric field due to point charge.

(B) Electric Field due to group of charges:

Suppose P is a point which is at $r_1, r_2, r_3, \ldots$ distances from the group of charges $q_1, q_2, q_3, \ldots$ The electric field at point P will be vector sum electric field produced by each charges at that point.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots\cdots = \frac{1}{4\pi\varepsilon}\frac{q_1}{r_1^3}\vec{r}_1 + \frac{1}{4\pi\varepsilon}\frac{q_2}{r_2^3}\vec{r}_2 + \frac{1}{4\pi\varepsilon}\frac{q_3}{r_3^3}\vec{r}_3 + \cdots\cdots = \frac{1}{4\pi\varepsilon}\sum_i \frac{q_i}{r_i^3}\vec{r}_i \qquad (4)$$

(C) Electric Field due to charge distribution:

Suppose dE is amount of electric field at point P due to charge dq in a charge distribution. Point P is at r distance from the charge dq.

$$dE = \frac{1}{4\pi\varepsilon}\frac{dq}{r^2} \qquad (5)$$

Total electric field at point P due to whole charge distribution can be obtained by integrating equation (5).

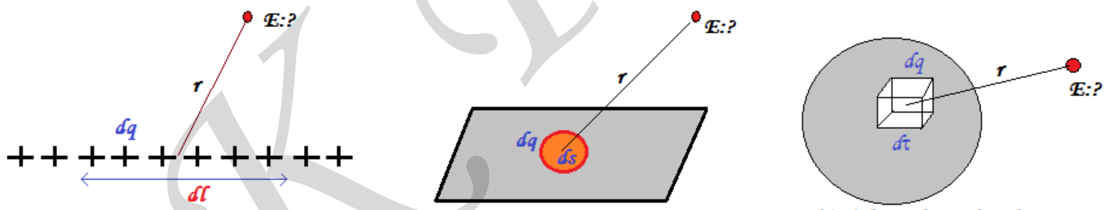
$$\boxed{E = \frac{1}{4\pi\varepsilon}\int\frac{dq}{r^2}} \qquad (6)$$

There are three types of charge distribution.

(i) ***Linear charge distribution:*** *If the charges are distributed over a line then it is called as linear charge distribution. The charge per unit length is called as linear charge density. It is denoted by λ. Let dq charges are distributed over a length dl then* $\lambda = \frac{dq}{dl}$.

(ii) ***Surface charge distribution:*** *If the charges are distributed over a surface then it is called as surface charge distribution. The charge per unit area is called as surface charge density. It is denoted by σ. Let dq charges are distributed over an area ds then* $\sigma = \frac{dq}{ds}$.

(iii) ***Volume charge distribution:*** *If the charges are distributed over a volume then it is called as volume charge distribution. The charge per unit area is called as volume charge density. It is denoted by ρ. Let dq charges are distributed over a volume dτ then* $\rho = \frac{dq}{d\tau}$.



(a) Linear charge distribution (b) Surface Charge distribution (c) Volume charge distribution

If the charge density remains unchanged throughout the charge distribution then the distribution is said to be uniform charge distribution otherwise it is called as non-uniform charge distribution.

Using equation (6), the electric field due to these three types of charge distribution can be written as:

$$E = \frac{1}{4\pi\varepsilon}\int\frac{\lambda dl}{r^2} \quad or \quad E = \frac{1}{4\pi\varepsilon}\int\frac{\sigma ds}{r^2} \quad or \quad E = \frac{1}{4\pi\varepsilon}\int\frac{\rho d\tau}{r^2}$$

Electric lines of forces:

Electric lines of forces are the imaginary lines in an electric filed which represents the path of test charge. The tangent on these lines at a point provides the direction of electric field at that point. The electric field will be large at the point where the electric lines of forces are dense. Similarly electric field will be weak at the point where the electric lines of forces are rarer.

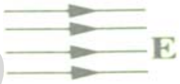
(a) ***Uniform electric field:*** *If the electric lines of forces are represented by parallel lines then electric field will be same at every point. Such electric lines of forces represent the uniform electric field.*

(b) ***Non-Uniform electric field:*** *If the electric lines of forces are linearly/curved diverging or converging then electric field will not be same at every point. Such electric lines of forces represent the non-uniform electric field. Thus position and time dependent electric field is called as non-uniform electric field. e.g.*

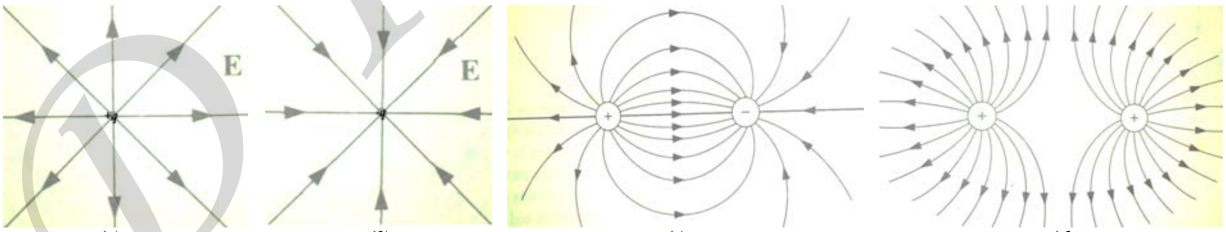
(1) *The path of test charge in electric field of positive point charge is linear along outward direction because the force acting on test charge is away from point charge. Hence the Electric lines of forces of positive point charge are outward linear. They are concentrated near the point charge and rarer away from the point charge thus positive point charge generates the diverging non-uniform electric field.*

(2) *The path of test charge in electric field of negative point charge is linear along inward direction because the force acting on test charge is towards the point charge. Hence the Electric lines of forces of negative point charge are inward linear. They are concentrated near the point charge and rarer away from the point charge thus negative point charge generates the converging non-uniform electric field.*

(3) *If the path of test charge in an electric field is converging or diverging curved lines then electric field will be same at every point. The electric field will be position and curvature dependent. Such electric lines of forces represent the non-uniform electric field.*



(i) Uniform electric field



(a) (b) (c) (d)

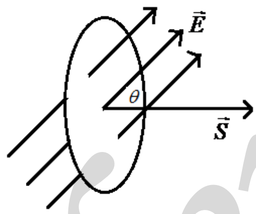
(ii) Non-uniform electric field: electric field due positive , negative and combination of charges

Note 1: The two electric lines of forces do not intersect to each other because direction of electric field can not be two at a single point.

Note 2: Parallel straight electric lines of forces represent the uniform electric field while converging or diverging curved/straight lines represent the non-uniform electric field

Electric Flux: The number of electric lines of forces passing through a surface is called as electric flux linked with surface. Mathematically, it is equal to product of component of electric field normal to the surface and area of surface. It is denoted by 'φ'.

Suppose the uniform electric field $\vec{E}$ represented by straight parallel electric lines of forces makes an angle θ with the area vector $\vec{S}$.



φ = component of $\vec{E}$ along $\vec{S}$ $\times$ area of surface

$$\varphi = E\, Cos\,\theta \cdot S$$

$$\boxed{\varphi = ES\, Cos\,\theta = \vec{E}\cdot\vec{S}} \qquad (1)$$

(i) When $\vec{E} \,||\, \vec{S}$ then $\theta = 0$ and hence $\varphi_{max} = E\,S$

(ii) When $\vec{E} \perp \vec{S}$ then $\theta = 90$ and hence $\varphi_{min} = 0$

(iii) Let $d\varphi$ electric flux linked with elementary area dS of a surface. Then,

$$d\varphi = \vec{E}\cdot d\vec{S} \qquad (2)$$

The total electric flux linked with surface can be obtained by integrating

$$\varphi = \int \vec{E}\cdot d\vec{S} \qquad (3)$$

If the surface is closed then total electric flux linked with surface can be written as;

$$\boxed{\varphi = \oint \vec{E}\cdot d\vec{S}} \qquad (4)$$

Gauss law: According to this law, "the net outward electric flux through closed surface is equal to the $1/\varepsilon_0$ times total charge enclosed by closed surface".

Let charges q_1, q_2, q_3, …, are enclosed by a closed surface. Then,

$$\text{Net outward electric flux} = \frac{1}{\varepsilon_0} \times \text{total charge enclosed by closed surface}$$

$$\varphi = \frac{1}{\varepsilon_0}\times(q_1 + q_2 + q_3 + \cdots) = \frac{1}{\varepsilon_0}\sum_i q_i = \frac{q}{\varepsilon_0}$$

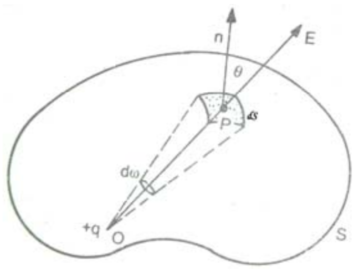
$$\boxed{\varphi = \oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}}$$

If charge enclosed by the closed surface is zero then the net outward electric flux will be zero.

Proof: Suppose total charge 'q' is enclosed by a closed surface and an elementary surface dS is at distance 'r' from the charge 'q'. If $\vec{E}$ electric field at distance 'r' from charge 'q' Then,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \qquad (1)$$

taking dot product with $d\vec{S}$ on the both sides of equation (1) we have,



$$\vec{E}\cdot d\vec{S} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}\cdot d\vec{S} \qquad (2)$$

Integrating equation (2) over a closed surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\oint \frac{\vec{r}\cdot d\vec{S}}{r^3} \qquad (3)$$

The projected area divided by r^2 is termed as solid angle (dω) subtended by dS at q.

$$\text{i.e. } d\omega = \frac{dS\,Cos\,\theta}{r^2} = \frac{r\,dS\,Cos\,\theta}{r^3} = \frac{\vec{r}\cdot d\vec{S}}{r^3} \qquad (4)$$

From equations (3) and (4), we have;

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\oint d\omega$$

As $\oint d\omega = 4\pi$ *thus,* $\oint \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\cdot 4\pi$

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \qquad (5)$$

Hence the law is proved.

Note: If q charge is at centre of an sphere of radius 'r' then net out outward electric flux,

$$\oint \vec{E}\cdot d\vec{S} = \oint \frac{1}{4\pi\varepsilon_0}\left(\frac{q\,\hat{r}}{r^2}\right)\cdot\left(r^2 Sin\theta\, d\theta\, d\phi\, \hat{r}\right) = \frac{q}{4\pi\varepsilon_0}\int_0^{\pi} Sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{q}{4\pi\varepsilon_0}\cdot 4\pi = \frac{q}{\varepsilon_0}$$

Divergence of $\vec{E}$ **:** *According to Gauss law, "the net outward electric flux through closed surface is equal to the* $1/\varepsilon_0$ *times total charge enclosed by closed surface".*

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \qquad (1)$$

If ρ *is charge density of distribution and dq charges are distributed over a volume* $d\tau$ *then,*

$$dq = \rho\, d\tau$$

$$\Rightarrow \quad q = \oint \rho\, d\tau \qquad (2)$$

From equations (1) and (2), we have;

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\oint \rho\, d\tau \qquad (3)$$

Using Gauss divergence theorem we can write that, $\oint \vec{E}\cdot d\vec{S} = \oint\left(\vec{\nabla}.\vec{E}\right)d\tau$,

hence *equations (3) becomes as;*

$$\oint\left(\vec{\nabla}.\vec{E}\right)d\tau = \oint \frac{\rho}{\varepsilon_0}\,d\tau \qquad (4)$$

Since equations (4) holds for any volume thus integrands must be equal.

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}} \qquad (5)$$

Equations (5) is called as differential form of Gauss law.

Note1: The closed surface considered in Gauss law is called as Gaussian surface. Symmetry is the essential criterion for the application of Gauss law. There are mainly three type of symmetry.

A) *Spherical symmetry: for a point charge, the Gaussian surface concentric spheres.*
B) *Cylindrical symmetry: for very long line charge distribution, the Gaussian surface concentric cylinders.*
C) *Plane symmetry: for very large surface charge distribution, the Gaussian pillbox is Gaussian closed surface.*

Note2: If charge is enclosed by closed surface is zero then net out word flux will be zero. This implies that divergence of electric field will also be zero. i.e.

$$\oint \vec{E}\cdot d\vec{S} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0$$

A vector field is solenoidal if its divergence vanishes. Since the divergence of electric field is zero, hence the electric field is said to be solenoidal.

Proof: The electric field due to point charge at a distance r is given by,

$$\vec{E} = \frac{1}{4\pi\varepsilon}\frac{q}{r^3}\vec{r}$$

Taking divergence on both sides we have,

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon}\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{q}{4\pi\varepsilon}\vec{\nabla}\cdot\left(r^{-3}\vec{r}\right); \qquad as\ \boxed{\vec{\nabla}.\phi\vec{A} = \left(\vec{\nabla}\phi\right).\vec{A} + \phi\left(\vec{\nabla}\cdot\vec{A}\right)}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon}\left[\left(\vec{\nabla}r^{-3}\right).\vec{r} + r^{-3}\left(\vec{\nabla}\cdot\vec{r}\right)\right] = \frac{q}{4\pi\varepsilon}\left[\left(-3r^{-5}\vec{r}\right).\vec{r} + r^{-3}(3)\right]$$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon}\left[-3r^{-5}\vec{r}.\vec{r} + 3r^{-3}\right] = \frac{q}{4\pi\varepsilon}\left[-3r^{-3} + 3r^{-3}\right] = 0$$

Hence $\vec{E}$ *is solenoidal for* $r \neq 0$.

'$\vec{E}$' ***irrotational and conservative field:***

- *Curl of a vector quantity is a measure of how much the vector curls around. If curl of a vector field vanishes then vector field is not rotational by it is irrotational field.*
- *For an irrotational field, the line integral of a vector field over a closed path is zero. This implies that the line integral of a vector field is path independent. Such vector field is called as conservative field.*
- *The electric field is irrotational and conservative field. i.e.*

$$\vec{\nabla}\times\vec{E} = 0$$

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad \Rightarrow \int \vec{E}\cdot d\vec{l} \text{ is path independent.}$$

Proof: (a) The electric field due to point charge at a distance r is given by,

$$\vec{E} = \frac{1}{4\pi\varepsilon}\frac{q}{r^3}\vec{r}$$

Taking curl on both sides we have,

$$\vec{\nabla}\times\vec{E} = \frac{q}{4\pi\varepsilon}\vec{\nabla}\times\frac{\vec{r}}{r^3} = \frac{q}{4\pi\varepsilon}\vec{\nabla}\times\left(r^{-3}\vec{r}\right); \quad as\ \boxed{\vec{\nabla}\times\phi\vec{A} = \left(\vec{\nabla}\phi\right)\times\vec{A} + \phi\left(\vec{\nabla}\times\vec{A}\right)}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon}\left[\left(\vec{\nabla}r^{-3}\right)\times\vec{r} + r^{-3}\left(\vec{\nabla}\times\vec{r}\right)\right] = \frac{q}{4\pi\varepsilon}\left[\left(-3r^{-5}\vec{r}\right)\times\vec{r} + r^{-3}(0)\right]$$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon}\left[-3r^{-5}\vec{r}\times\vec{r} + 0\right] = \frac{q}{4\pi\varepsilon}[0+0] = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad (1)$$

Hence $\vec{E}$ is irrotational field. We know that $\vec{E}$ follows the superposition principle.

i.e. $\qquad \vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots\cdots$

So, $\qquad \vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{E}_1 + \vec{\nabla}\times\vec{E}_2 + \vec{\nabla}\times\vec{E}_3 + \cdots\cdots = 0 \qquad (2)$

Thus $\vec{E}$ is irrotational field for any static charge distribution, i.e. there is no matter where the charge is located or it is distributed.

(b) Taking volume integral of equation (1) over a closed path we have,

$$\oint_v \left(\vec{\nabla}\times\vec{E}\right)dv = 0 \qquad (3)$$

From stokes theorem, the volume integral can be converted into line integral with following formula.

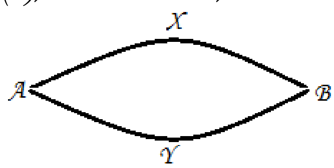
$$\oint_v \left(\vec{\nabla}\times\vec{E}\right)dv = \oint_l \vec{E}.d\vec{l} \qquad (4)$$

From equation (3) and (4), we can write,

$$\oint_l \vec{E}.d\vec{l} = 0 \qquad (5)$$

If closed path is AXBYA then from equation (5), we can write,

$$\int_A^B \vec{E}.d\vec{l} + \int_B^A \vec{E}.d\vec{l} = 0$$



$$\int_A^B \vec{E}.d\vec{l} = -\int_B^A \vec{E}.d\vec{l}$$

$$\int_{path\ AXB} \vec{E}.d\vec{l} = \int_{path\ AYB} \vec{E}.d\vec{l} \qquad (6)$$

Equation (6) indicates that the line integral of a electric field is path independent. Thus it is conservative field.

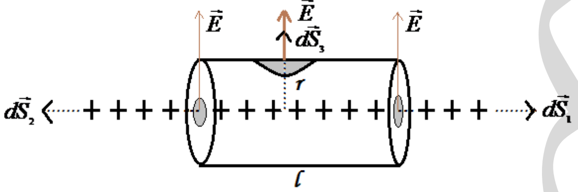
Application of Gauss Law

1. Electric field due to infinite linear charge distribution

Suppose charge are distributed along a line, whose linear charge density is 'λ'. If q charge is distributed on the length l then,

$$q=\lambda l \tag{1}$$

If the charge is positive then the electric field will be away from the line. Let Electric field at distance 'r' due to this charge distribution is $\vec{E}$. The Gaussian surface for this charge distribution is a cylinder of radius 'r' and length 'l' whose axis contains charge distribution.



Applying Gauss law to this Gaussian surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \tag{2}$$

Since the Gaussian surface is composed of three surfaces s_1, s_2 and s_3. Thus equation (2) can written as,

$$\int_{s_1} \vec{E}\cdot d\vec{S}_1 + \int_{s_2} \vec{E}\cdot d\vec{S}_2 + \int_{s_3} \vec{E}\cdot d\vec{S}_3 = \frac{q}{\varepsilon_0}$$

$$\int_{s_1} E\, dS_1 Cos90 + \int_{s_2} E\, dS_2 Cos90 + \int_{s_3} E\, dS_3 Cos0 = \frac{q}{\varepsilon_0}$$

$$0+0+\int_{s_3} E\, dS_3 = \frac{q}{\varepsilon_0}$$

$$\int_{s_3} E\, dS_3 = \frac{q}{\varepsilon_0} \quad \Rightarrow \quad E\int_{s_3} dS_3 = \frac{q}{\varepsilon_0}$$

$$E 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}}$$

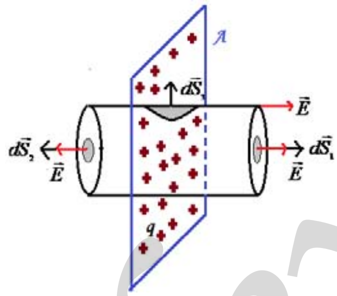
2. Electric field due to uniformly charged plane sheet

Method 1: *Consider a plane sheet of area A is uniformly charged with charge 'q'. If surface charge density is 'σ' then charge on plane sheet will be written as;*

$$q=\sigma A \tag{1}$$

If the charge is positive then the electric field will be away from the surface and for negative it points towards plane. Since charges are distributed on plane thus cylindrical Gaussian surface *(pill-box) shall be the Gaussian closed surface for it whose half portion lies on one side of the plane and rest on the other side.*

Applying Gauss law to this Gaussian surface,



$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \qquad (2)$$

Since the Gaussian surface is composed of three surfaces s_1, s_2 and s_3. Thus equation (2) can written as,

$$\int_{s_1} \vec{E}\cdot d\vec{S}_1 + \int_{s_2} \vec{E}\cdot d\vec{S}_2 + \int_{s_3} \vec{E}\cdot d\vec{S}_3 = \frac{q}{\varepsilon_0}$$

$$\int_{s_1} E\, dS_1 Cos0 + \int_{s_2} E\, dS_2 Cos0 + \int_{s_3} E\, dS_3 Cos90 = \frac{q}{\varepsilon_0}$$

$$\int_{s_1} E\, dS_1 + \int_{s_2} E\, dS_2 + 0 = \frac{q}{\varepsilon_0}$$

$$E\int_{s_1} dS_1 + E\int_{s_2} dS_2 = \frac{q}{\varepsilon_0}$$

$$EA + EA = \frac{q}{\varepsilon_0}$$

$$2EA = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

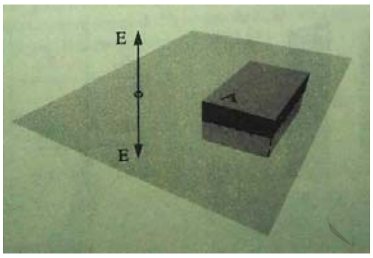
Method 2: *Consider a plane sheet of area A is uniformly charged with charge 'q'. If surface charge density is 'σ' then charge on plane sheet will be written as;*

$$q=\sigma A \qquad (1)$$

If the charge is positive then the electric field will be away from the surface and for negative it points towards plane. Since charges are distributed on plane thus pill-box shall be the Gaussian closed surface for it whose half portion lies on one side of the plane and rest on the other side. The electric flux through half pill-box will be EA.

Applying Gauss law to this Gaussian surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$



$$net\ out\ ward\ \ flux\ through\ pill\text{-}box = \frac{q}{\varepsilon_0}$$

$$2\times flux\ through\ half\ \ pill\text{-}box = \frac{q}{\varepsilon_0}$$

$$2EA = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

3. Electric field due to uniformly charged sphere

Consider a sphere of radius 'R' is uniformly charged with charge 'q'. If volume charge density is 'ρ' then charge on plane sheet will be written as;

$$q = \rho v = \rho\frac{4\pi}{3}R^3 \qquad (1)$$

Since charges are distributed within the sphere thus concentric sphere shall be the Gaussian surface for it. For the knowledge of variation of electric field with radial distance we have to determine the electric field at three different point's i.e. external, surface and internal point.

(i) *At external point:* Let the electric field at external point (at distance r (>R) from the centre of charge distribution) is $\vec{E}_{ex}$. The Gaussian surface for it is concentric sphere of radius 'r' and external point lies on the surface of it. If charge distribution is of positive charge then field will be normal away from this Gaussian surface.

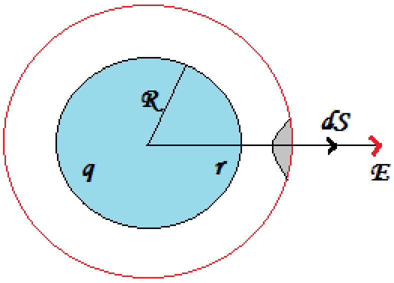
Applying Gauss law to this Gaussian surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$\because \vec{E}\ //\ d\vec{S}$ and total charge enclosed by Gaussian surface is q thus,

$$\oint E_{ex}\ dS\cos 0 = \frac{q}{\varepsilon_0}$$

$$E_{ex}\oint dS = \frac{q}{\varepsilon_0}$$



$$E_{ex}\,4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\boxed{E_{ex} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$$

$$\Rightarrow E_{ex} \propto \frac{1}{r^2} \qquad (2a)$$

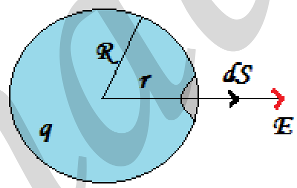
Using equation (1), electric field can be expressed in terms of charge density.

$$E_{ex} = \frac{1}{4\pi\varepsilon_0}\frac{\rho\frac{4\pi}{3}R^3}{r^2}$$

$$\boxed{E_{ex} = \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}} \quad \Rightarrow \quad E_{ex} \propto \frac{1}{r^2} \qquad (2b)$$

(ii) At surface point: Let the electric field at the surface of charge distribution ($r = R$) is $\vec{E}_s$. The Gaussian surface for it is concentric sphere of radius 'R'. In this situation, Gaussian surface also encloses all the charges distributed in the spherical charge distribution. Thus surface electric field can be obtained by putting $r=R$ in equation (2).

$$\boxed{E_s = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}}$$

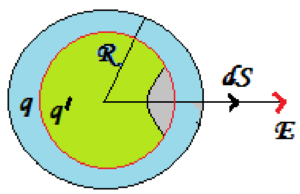


and $$\boxed{E_s = \frac{\rho R}{3\varepsilon_0}} \qquad (3)$$

(iii) At internal point: Let the electric field at internal point (at distance r ($<R$) from the centre of charge distribution) is $\vec{E}_{in}$. The Gaussian surface for it is concentric sphere of radius 'r'. In this situation, Gaussian surface does not enclose all the charges distributed in the spherical charge distribution but it surrounds only a fraction of total charge say it is q'.

Applying Gauss law to this Gaussian surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\times \text{charge enclosed by Gaussian surface}$$



$\because \vec{E} \,//\, d\vec{S}$ and total charge enclosed by Gaussian surface is q' thus,

$$\oint E_{in}\, dS \cos 0 = \frac{q'}{\varepsilon_0}$$

$$E_{in}\oint dS = \frac{q'}{\varepsilon_0}$$

$$E_{in}\, 4\pi r^2 = \frac{q'}{\varepsilon_0}$$

$$E_{in} = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2} \qquad (4)$$

Since charge density is constant within the Gaussian surface and for the spherical charge distribution thus,
Charge density of charge distribution= Charge density within Gaussian surface

$$\frac{q}{\frac{4\pi}{3}R^3} = \frac{q'}{\frac{4\pi}{3}r^3}$$

$$\Rightarrow \qquad q' = q\frac{r^3}{R^3} \qquad (5)$$

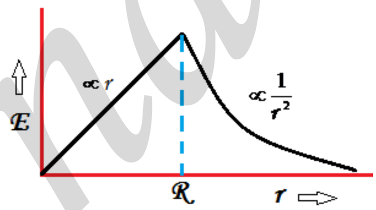
Using equation (4) and (5) we have,

$$\boxed{E_{in} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}} \qquad \Rightarrow \qquad E_{in} \propto r \qquad (6a)$$

Using equation (1), the field can be found in terms of charge density.

$$\boxed{E_{in} = \frac{\rho r}{3\varepsilon_0}} \qquad \Rightarrow \qquad E_{in} \propto r \qquad (6b)$$

From equations (2), (3) and (6) it is clear that the electric field is directly proportional to 'r' within the charge distribution and inversely proportional to r^2 outside the spherical charge distribution while it is maximum at the surface.



4. Electric field due to charged spherical shell

Consider a spherical shell of radius 'R' is given to charge 'q'. For the knowledge of variation of electric field with radial distance we have to determine the electric field at three different point's i.e. external, surface and internal point.

(iv) At external point: Let the electric field at external point (at distance r (>R) from the centre of charge distribution) is $\vec{E}_{ex}$. The Gaussian surface for it is concentric sphere of radius 'r' and external point lies on the surface of it. If charge distribution is of positive charge then field will be normal away from this Gaussian surface.

Applying Gauss law to this Gaussian surface,

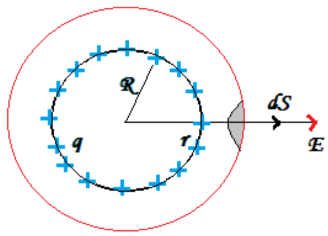
$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$\because \vec{E} \,//\, d\vec{S}$ and total charge enclosed by Gaussian surface is q thus,

$$\oint E_{ex}\, dS \cos 0 = \frac{q}{\varepsilon_0} \qquad \Rightarrow \qquad E_{ex}\oint dS = \frac{q}{\varepsilon_0} \qquad \Rightarrow E_{ex}\, 4\pi r^2 = \frac{q}{\varepsilon_0}$$

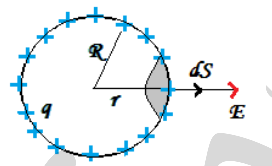
$$\boxed{E_{ex} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$$

$$\Rightarrow E_{ex} \propto \frac{1}{r^2} \qquad (2)$$



(v) *At surface point:* *Let the electric field at the surface of charge distribution (r =R) is* $\vec{E}_s$. *The Gaussian surface for it is concentric sphere of radius 'R'. In this situation, Gaussian surface also encloses all the charges distributed in the spherical shell. Thus surface electric field can be obtained by putting r=R in equation (2).*

$$\boxed{E_s = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}} \qquad (3)$$



(vi) *At internal point:* *Let the electric field at internal point (at distance r (<R) from the centre of charge distribution) is* $\vec{E}_{in}$. *The Gaussian surface for it is concentric sphere of radius 'r'. In this situation, Gaussian surface does not enclose any charges.*

Applying Gauss law to this Gaussian surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0} \times \textit{charge enclosed by Gaussian surface}$$

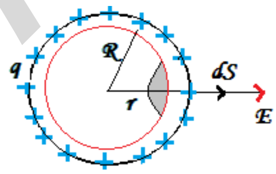
$\because \vec{E} \,//\, d\vec{S}$ and total charge enclosed by Gaussian surface is zero thus,

$$\oint E_{in}\, dS \cos 0 = 0$$

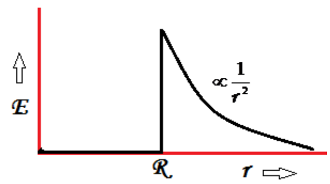
$$E_{in} \oint dS = 0$$

$$E_{in}\, 4\pi r^2 = 0$$

$$E_{in} = 0 \qquad (4)$$



From equations (2), (3) and (4) it is clear that the electric field is zero within the spherical shell and inversely proportional to r^2 *outside the spherical shell while it is maximum at the surface.*



5. Electric field due to charged conductor of any shape

The charges are free to move in a conductor even under influence of very small electric field. Hence, when a charge is given to conductor, then they move until they find the position in which no net force act on them. Due to this, interior of conductor becomes depleted of charge carriers and all the charges come on the surface of conductor.

Since no charges are present at interior of conductor thus electric filed is zero at interior of conductor. i.e. $E_{in} = 0$

Let the surface charge density of the charged conductor is σ. Suppose that P is point lying just outside the surface of conductor and electric field at this point is to be found. Consider a small cylindrical Gaussian surface CD having one base C at P and other base D lying inside the conductor.

Applying Gauss law to this cylindrical Gaussian surface,

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\times \text{charge enclosed by Gaussian surface}$$

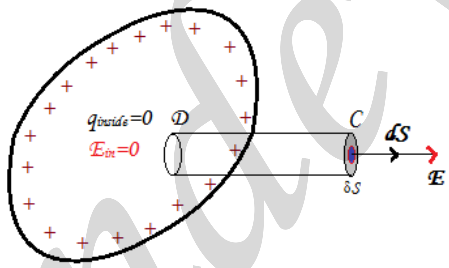
∵ *There are three surface in the cylinder C, D and CD (curved surface). The electric flux through surface D is zero as no charge present at inside the conductor. Similarly the flux through curved surface CD will also be zero as electric field is normal to area vector for this surface. Thus flux will be only through surface C as* $\vec{E} \,//\, d\vec{S}$.

$$\int_{surface\,C} \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\times \text{charge enclosed by Gaussian surface}$$

$$\int_{surface\,C} E\, dS\, Cos\, 0 = \frac{1}{\varepsilon_0}\sigma\,\delta S$$

$$E \int_{surface\,C} dS = \frac{1}{\varepsilon_0}\sigma\,\delta S \quad E\,\delta S = \frac{1}{\varepsilon_0}\sigma\,\delta S$$

$$E = \frac{\sigma}{\varepsilon_0}$$



Thus electric field at any point close to the surface of charged conductor is 1/ε_0 times the surface charge density.

Que1: A space is filled with charge whose charge density varies according to the law $\rho=\rho_0/r$. Here ρ_0 is constant and r is distance from the origin of co-ordinates. Find the electric field as function of position vector.

Ans: Consider a spherical Gaussian surface of radius r with the centre of origin. Applying Gauss law,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\oint E\, dS\, Cos0 = \frac{1}{\varepsilon_0}\oint \rho\, dv$$

$$\oint E\, dS\, Cos0 = \frac{1}{\varepsilon_0}\int \frac{\rho_0}{r}4\pi r^2 dr$$

$$E 4\pi r^2 = \frac{\rho_0 4\pi}{\varepsilon_0}\int r\, dr$$

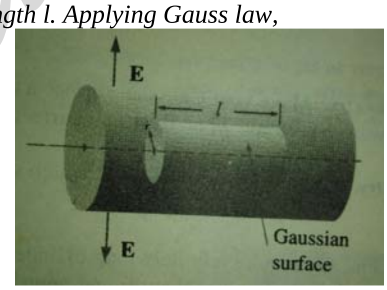
$$E 4\pi r^2 = \frac{\rho_0 4\pi}{\varepsilon_0}\frac{r^2}{2}$$

$$E = \frac{\rho_0}{2\varepsilon_0}$$

$$\vec{E} = \frac{\rho_0}{2\varepsilon_0}\hat{r}$$

Que2: A long cylinder is charged whose charge density varies according to the law $\rho=kr$. Here r is radius of cylinder. Find the electric field inside the long cylinder.

Ans: Consider a cylindrical Gaussian surface of radius r and length l. Applying Gauss law,



$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\oint E\, dS\, Cos0 = \frac{1}{\varepsilon_0}\oint \rho\, dv$$

$$\int_{curved\ surface} E\, dS\, Cos0 = \frac{1}{\varepsilon_0}\iiint (kr)\, r dr\, d\theta\, dz$$

$$E \int_{curved\ surface} ds = \frac{k}{\varepsilon_0}\int_0^r r^2 dr \int_0^{2\pi} d\theta \int_0^{l} dz$$

$$E 2\pi r l = \frac{k}{\varepsilon_0}\frac{r^3}{3}2\pi l$$

$$E = \frac{k}{3\varepsilon_0}r^2$$

$$\vec{E} = \frac{k}{3\varepsilon_0}r^2\hat{r}$$

Que3: A charge density of hallow spherical shell of inner and outer radii a and b respectively varies as $\rho = k/r^2$ in the region a≤r≤b. Find the electric field in the regions (i) r<a (ii) a≤r<b and (iii)r>b

Ans: According to Gauss law,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

(i) *In region r<a: Since no charge is enclosed in this region thus*

$$\oint \vec{E}\cdot d\vec{S} = 0$$

$E=0$

(ii) *In region a≤r<b: consider a spherical Gaussian surface lying inside the spherical shell. In this situation charges lying between radii 'a' to 'r' are responsible to the electric field. Applying Gauss law to this surface.*

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\oint E\, dS\, Cos0 = \frac{1}{\varepsilon_0}\oint \rho\, dv$$

$$E 4\pi r^2 = \frac{1}{\varepsilon_0}\int_a^r \frac{k}{r^2} 4\pi r^2 dr$$

$$E = \frac{k}{\varepsilon_0}\frac{(r-a)}{r^2}$$

(iii) *In region r>b: consider a spherical Gaussian surface lying outside the spherical shell. In this situation all the charges existing between radii 'a' and 'b' are responsible to the electric field. Applying Gauss law to this surface.*

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\oint E\, dS\, Cos0 = \frac{1}{\varepsilon_0}\oint \rho\, dv$$

$$E 4\pi r^2 = \frac{1}{\varepsilon_0}\int_a^b \frac{k}{r^2} 4\pi r^2 dr$$

$$E = \frac{k}{\varepsilon_0}\frac{(b-a)}{r^2}$$

Que4: The electric field due to spherical charge distribution varies radial as $E = ar + br^2$. Find the total charge enclosed within the radius of r_0.

*Ans: **Method 1**: Given that,* $E = ar + br^2$

So, at $r=r_0$, $E_0 = ar_0 + br_0^2$

According to Gauss law,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$q = \varepsilon_0 \oint \vec{E}_0 \cdot d\vec{S}$$

$$q = \varepsilon_0 \oint E_0\, dS\, cos0$$

$$q = \varepsilon_0 E_0 \oint dS$$

$$q = \varepsilon_0 E_0 \int_0^{r_0} ds$$

$$q = \varepsilon_0 (ar_0 + br_0^2) 4\pi r_0^2$$

$$\boxed{q = 4\pi\varepsilon_0 (a + br_0) r_0^3}$$

Method 2: Given that,

$$E = ar + br^2$$

$$\vec{E} = (ar + br^2)\hat{r}$$

$$\vec{E} = (ar + br^2)\frac{\vec{r}}{r}$$

$$\vec{E} = (a\vec{r} + br\vec{r})$$

According to *Differential form* Gauss law,

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\rho = \varepsilon_0 \vec{\nabla}.\vec{E}$$

$$\rho = \varepsilon_0 \vec{\nabla}.(a\vec{r} + br\vec{r})$$

$$\rho = \varepsilon_0 (a\vec{\nabla}.\vec{r} + b\vec{\nabla}.r\vec{r})$$

$$\rho = \varepsilon_0 (3a + b(r\vec{\nabla}.\vec{r} + \vec{r}.\vec{\nabla}r))$$

$$\rho = \varepsilon_0 (3a + b(r3 + \vec{r}.\frac{\vec{r}}{r}))$$

$$\rho = \varepsilon_0 (3a + b(3r + \frac{\vec{r}.\vec{r}}{r}))$$

$$\rho = \varepsilon_0 (3a + b(3r + r))$$

$$\rho = \varepsilon_0 (3a + 4br)$$

$$\because\ q = \int \rho\, dv$$

$$\therefore\ q = \oint \{\varepsilon_0 (3a + 4br)\} dv$$

$$\Rightarrow q = \varepsilon_0 \int_0^{r_0} \{3a + 4br\} 4\pi r^2\, dr$$

$$\Rightarrow q = \varepsilon_0 \int_0^{r_0} \{3a + 4br\} 4\pi r^2\, dr$$

$$\Rightarrow q = 4\pi\varepsilon_0 \int_0^{r_0} \{3ar^2 + 4br^3\} dr$$

$$\Rightarrow q = 4\pi\varepsilon_0 \left(3a\frac{r_0^3}{3} + 4b\frac{r_0^4}{4}\right)$$

$$\Rightarrow \boxed{q = 4\pi\varepsilon_0 (a + br_0) r_0^3}$$